Corrigendum: Optimal Control of Semilinear Elliptic Equations with Pointwise Constraints on the Gradient of the State

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Contrary to the claim in our paper [1], equation (6.11) cannot be used to define the normal derivative $\partial_{\nu_A^*}(p)$ for elements $p \in V^t(\Omega)$ because the term $\langle A^*p, z \rangle_\Omega$ depends on the representation of A^*p as an element of $C_0^1(\Omega)'$ for elements $z \in W^{2,t'}(\Omega)$; see Lemma 2. Therefore Theorem 9 is not properly formulated and then the solution of the adjoint state equation (6.7) must be intended in a variational sense, as for the classical Neumann problem with a datum belonging to $(H^1(\Omega))'$. More precisely

Definition. Given $\mu \in M(\bar{\Omega})^n$, $y \in L^\infty(\Omega)$, and $t \in (1, n/(n-1))$, we say that an element $p \in V^t(\Omega)$ is a solution of the problem

$$(PN) \quad \begin{cases} A^*p + \varphi'(y)p = -\operatorname{div} \mu_\Omega & \text{in } \Omega, \\ \partial_{\nu_A^*} p = -\sum_{j=1}^n \partial_{t_j} \mu_\Gamma^j & \text{on } \Gamma, \end{cases}$$

with t_j defined as in Theorem 8, if and only if p satisfies

$$\begin{aligned} & \int_\Omega p(Az + \varphi'(y)z) dx + \langle y(p), \partial_{\nu_A^*} z \rangle \\ &= \sum_{j=1}^n \left(\int_\Omega \partial_{x_j} z d\mu_\Omega^j + \int_\Gamma \partial_{t_j} z d\mu_\Gamma^j \right), \quad \forall z \in W^{2,t'}(\Omega). \end{aligned}$$

Existence and uniqueness of such a solution is established in the first part of the proof of Theorem 10.

Nevertheless, let us remark that the definition of $\gamma(p)$ given through equation (4.4) is correct because the term $\langle A^*p, z \rangle_\Omega$ is independent of the representation of A^*p for every $z \in W^{2,1'}(\Omega) \cap W_0^{1,1'}(\Omega)$, as proved below.

Lemma. *Let T be an element of $C_0^1(\Omega)'$. Then, for each $z \in C_0(\Omega) \cap C^1(\bar{\Omega})$, the expression*

$$\langle T, z \rangle_\Omega = \int_\Omega z \, d\mu_0 + \sum_{j=1}^n \int_\Omega \partial_{x_j} z \, d\mu_j$$

is independent of the representation of T :

$$T = \mu_0 - \sum_{j=1}^n \partial_{x_j} \mu_j \quad \text{in } D'(\Omega),$$

with $\mu_j \in M(\Omega)$, $j = 0, 1, \dots, n$.

Proof. Given $\mu_j \in M(\Omega)$, $j = 0, 1, \dots, n$, such that

$$\int_\Omega z \, d\mu_0 + \sum_{j=1}^n \int_\Omega \partial_{x_j} z \, d\mu_j = 0, \quad \forall z \in C_0^1(\Omega),$$

we must prove that the same relation holds for every $z \in C_0(\Omega) \cap C^1(\bar{\Omega})$. For this purpose, we use Hopf's method [2]: given $\varepsilon > 0$ there exist functions $\theta_\varepsilon \in C^2(\bar{\Omega})$ verifying the following conditions:

- $0 \leq \theta_\varepsilon(x) \leq 1$, $\forall x \in \bar{\Omega}$.
- $\theta_\varepsilon(x) = 1$, $\forall x \in \Gamma_\varepsilon$, where Γ_ε is a neighborhood of Γ .
- $\theta_\varepsilon(x) = 0$ if $d(x, \Gamma) \geq 2 \exp(-1/\varepsilon)$.
- $|\partial_{x_j} \theta_\varepsilon(x)| \leq \varepsilon/d(x, \Gamma)$, $\forall x \in \Omega$, $\forall j = 1, \dots, n$.

For every $z \in C_0(\Omega) \cap C^1(\bar{\Omega})$, let us consider the sequence $\{z_\varepsilon\} \subset C_0^1(\Omega)$ defined by

$$z_\varepsilon(x) = z(x)(1 - \theta_\varepsilon(x)).$$

It is immediate to verify that

$$|z_\varepsilon(x)| \leq |z(x)| \quad \text{and} \quad z_\varepsilon(x) \rightarrow z(x), \quad \forall x \in \bar{\Omega}.$$

Moreover, using the mean value theorem we can derive the existence of a positive constant C such that

$$|z(x)| \leq Cd(x, \Gamma), \quad \forall x \in \Omega.$$

Actually it is enough to observe that, for every $x \in \Omega$ and an arbitrary $\bar{x} \in \Gamma$,

$$|z(x)| = |z(x) - z(\bar{x})| \leq C|x - \bar{x}|.$$

Therefore, we have

$$|\partial_{x_j} z_\varepsilon(x)| \leq |\partial_{x_j} z(x)| + C\varepsilon \quad \text{and} \quad \partial_{x_j} z_\varepsilon(x) \rightarrow \partial_{x_j} z(x), \quad \forall x \in \Omega.$$

We conclude the proof taking into account these relations and applying the dominated convergence theorem. $\square$

References

1. Casas E, Fernández LA (1993) Optimal control of semilinear elliptic equations with pointwise constraints on the gradient of the state. *Appl Math Optim* 27:35–56
2. Hopf E, (1957) On nonlinear partial differential equations. In: *Lecture Series of the Symposium on Partial Differential Equations*. The University of Kansas, pp 1–29