ERRATUM

SIMPLY INVARIANT SUBSPACES OF H^2 OF SOME MULTIPLY CONNECTED REGIONS.

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Lemma 2.3 of the paper is false the way it has been stated, the proof we gave is wrong, and at this point we do not know how to prove a true statement of similar generality. The following Theorem, however, will close the gap in the proofs of the rest of the paper.

We would like to thank Professor Donald Sarason for bringing the error to our attention. We are very sorry for any inconveniences that our error might have caused.

We shall use the terminology as in the paper.

Lemma 2.3, its proof, and the remark following it should be deleted. The following Theorem should be inserted in the middle of page 136 at the point where the reference to Lemma 2.3 is made. Lemmas 3.1 and 3.2 were proved without use of Lemma 2.3.

THEOREM. *Let $\mathcal{M}$ be a simply invariant subspace of $H^2(\mathbb{A}_R)$, let $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$.*

Then for every $f \in \mathcal{M}$

$$\|f\|_{H^2(\mathbb{A}_R)}^2 = \left\| \frac{f}{\varphi} \right\|_{H^2(\mathbb{D})}^2 + (R^2 - 1) \int_{|\lambda|=1} \int_{|z|=R} \left| \frac{f(z) - \frac{f}{\varphi}(\lambda)\varphi(z)}{z - \lambda} \right|^2 |dz| \frac{|d\lambda|}{2\pi}.$$

Note that this theorem gives the conclusion of Lemma 2.3 for every simply invariant subspace of $H^2(\mathbb{A}_R)$. In fact, this is formula (3.1) in the paper.

Proof. We start with the formula on the bottom of page 133 which in our case is equivalent to the formula in Lemma 2.2, where $T = M_z|_{\mathcal{M}}$. Recall that L was defined to be the left inverse of T with $\varphi \in \ker L$, $Lf(z) = \frac{f(z) - \langle f, \varphi \rangle \varphi(z)}{z}$. For $\lambda \in \mathbb{D}$ we set $L_\lambda =$

$L(1-\lambda L)^{-1}$. On page 133 it was shown that $L_\lambda f(z) = \frac{f(z) - \langle (1-\lambda L)^{-1} f, \varphi \rangle \varphi(z)}{z-\lambda}$, and on page 136 we remarked that the pseudocontinuation of $\frac{f}{\varphi}$ to $\mathbb{D}$ equals $\frac{f}{\varphi}(\lambda) = \langle (1-\lambda L)^{-1} f, \varphi \rangle$. Thus, we obtain for any $r < 1$ the identity

$$\|f\|^2 = \left\| \frac{f}{\varphi} \right\|_{H^2(r\mathbb{D})}^2 + \int_{|\lambda|=r} \|DL_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} + (1-r^2) \int_{|\lambda|=r} \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r},$$

where D denotes the positive operator with $D^2 = M_z^* M_z - I$. Note that the limit as $r \rightarrow 1^-$ of the first two summands exists, and it equals the expression in the theorem.

Thus, we know that $f/\varphi \in H^2(\mathbb{D})$ and $(1-r^2) \int_{|\lambda|=r} \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} \leq \|f\|^2$ for each $r < 1$, and we must show that $\limsup_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} = 0$.

We will start out by proving that for any $g \in H^\infty(\mathbb{D})$ such that $g(f/\varphi) \in H^\infty(\mathbb{D})$ $\limsup_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} = 0$. This will imply for example that the Theorem is true, if f/φ is bounded on the unit circle. However, we shall need this fact in its full strength later on.

From the particular form of $L_\lambda f(z)$ it is clear that

$$\lim_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 \int_{|z|=R} |L_\lambda f(z)|^2 |dz| \frac{|d\lambda|}{2\pi r} = 0,$$

and it suffices to show that $\lim_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 \int_{|z|=1} |L_\lambda f(z)|^2 |dz| \frac{|d\lambda|}{2\pi r} = 0$. We note that

$$\begin{aligned} & (1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 |L_\lambda f(z)|^2 \frac{|d\lambda|}{2\pi r} \\ &= \int_{|\xi|=1} \frac{1-r^2}{|1-zr\xi|^2} \left| \left(\left(g \frac{f}{\varphi} \right)(z) - \left(g \frac{f}{\varphi} \right)(r\xi) \right) \varphi(z) - (g(z) - g(r\xi))f(z) \right|^2 |d\xi| \\ &\leq 4 \left(\left\| g \frac{f}{\varphi} \right\|_{H^\infty}^2 |\varphi(z)|^2 + \|g\|_{H^\infty}^2 |f(z)|^2 \right). \end{aligned}$$

By expanding the term in the middle we see that the limit as $r \rightarrow 1$ is 0 for a.e. $|z| = 1$, thus we can apply the dominated convergence theorem and Fubini's theorem to conclude that

$$\lim_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 \int_{|z|=1} |L_\lambda f(z)|^2 |dz| \frac{|d\lambda|}{2\pi r} = 0.$$

Next we note that the operator $L : \mathcal{M} \rightarrow \mathcal{M}$ is a completely nonunitary contraction. In fact, $Lf = (f - \langle f, \varphi \rangle \varphi)/z$, hence $\|Lf\|^2 \leq \|zLf\|^2 = \|f - \langle f, \varphi \rangle \varphi\|^2 = \|f\|^2 - |\langle f, \varphi \rangle|^2 \leq \|f\|^2$. If $\mathcal{N} \subseteq \mathcal{M}$ is a reducing subspace for L such that $L|_{\mathcal{N}}$ is unitary, then $\|Lf\| = \|f\|$ for all $f \in \mathcal{N}$. This implies that $\varphi \perp f$ and $zLf = f$ for each $f \in \mathcal{N}$, and hence $M_z = (L|_{\mathcal{N}})^{-1}$ is unitary on $\mathcal{N}$. We see that every $f \in \mathcal{N}$ is identically zero on $\{|z| = R\}$, hence $\mathcal{N} = \{0\}$.

Let $g \in H^\infty(\mathbb{D})$ and write $\tilde{g}$ for the function defined by $\tilde{g}(z) = \overline{g(\bar{z})}$. Then by the Sz. Nagy-Foias functional calculus the operators $\tilde{g}(L)$ and $\tilde{g}(r^2 L)$ are well defined for each $r < 1$. A calculation with power series coefficients shows that $\int_{|\lambda|=r} g(\lambda) \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} = \int_{|\lambda|=r} \langle L_\lambda f, L_\lambda \tilde{g}(r^2 L) f \rangle \frac{|d\lambda|}{2\pi r}$. Then

$$\begin{aligned} & (1-r^2) \int_{|\lambda|=r} \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} \\ &= (1-r^2) \int_{|\lambda|=r} (1-g(\lambda)) \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} + (1-r^2) \int_{|\lambda|=r} g(\lambda) \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} \\ &\leq \|f\| \|(I - \tilde{g}(r^2 L))f\| + \|f\| \left((1-r^2) \int_{|\lambda|=r} |g(\lambda)|^2 \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} \right)^{1/2} \end{aligned}$$

Since $\tilde{g}(r^2 L)$ converges to $\tilde{g}(L)$ in the strong operator topology we obtain for any $g \in H^\infty(\mathbb{D})$ such that $g(f/\varphi) \in H^\infty(\mathbb{D})$

$$\limsup_{r \rightarrow 1^-} (1-r^2) \int_{|\lambda|=r} \|L_\lambda f\|^2 \frac{|d\lambda|}{2\pi r} \leq \|f\| \|(I - \tilde{g}(L))f\|.$$

Thus, the proof can be concluded by the construction of a sequence $g_n \in H^\infty(\mathbb{D})$, $g_n(f/\varphi) \in H^\infty(\mathbb{D})$, such that $\|g_n\|_{H^\infty} \leq 1$ and $g_n(z) \rightarrow 1$ a.e. $|z| = 1$, because for such a sequence the properties of the Sz. Nagy-Foias functional calculus will imply that $\tilde{g}_n(L) \rightarrow I$ in the strong operator topology.

One way to construct such a sequence is as follows: we start by defining outer functions ψ_n by $|\psi_n(e^{it})| = 1$ if $|f/\varphi(e^{it})| < n$ and $|\psi_n(e^{it})| = n|\varphi/f(e^{it})|$ if $|f/\varphi(e^{it})| \geq n$. Then $\|\psi_n\|_\infty \leq 1$, $\|\psi_n f/\varphi\|_\infty \leq n$, and $\psi_n(z) \rightarrow 1$ uniformly on compact subsets of the unit disc $\mathbb{D}$. Thus, $\psi_n \rightarrow 1$ weakly in $H^2(\mathbb{D})$, and some sequence of finite convex combinations of ψ_n will converge to 1 in the $H^2(\mathbb{D})$ -norm. Then, a subsequence of these convex combinations will be the sequence g_n that converges to 1 for a.e. $|z| = 1$, and satisfies $\|g_n\|_\infty \leq 1$ and $g_n(f/\varphi) \in H^\infty(\mathbb{D})$.

This concludes the proof of the Theorem. ■

In our paper we give one more reference to Lemma 2.3, and that is on page 139, where we claim that the analogue of the Theorem that we just proved is true for 1-connected regions U that have the property that there is a conformal map ϕ of a neighborhood of U that takes the interior G of the bounded component of the complement of U to the open unit disc.

The proof of that statement follows the proof of the above theorem, where we substitute the operator of multiplication by ϕ , M_ϕ , for T , and we use the left inverse $Lf = (f - \langle f, \varphi \rangle \varphi)/\phi$, where $\varphi \in \mathcal{M} \ominus (z - \lambda_0)\mathcal{M}$, $\|\varphi\| = 1$, $\phi(\lambda_0) = 0$. We will show how one gets started.

If μ is a point in the unit disc and $\lambda \in G$ such that $\phi(\lambda) = \mu$, then one verifies that for every $f \in \mathcal{M}$ and $h \in \mathcal{M}^\perp$

$$\langle (1 - \mu L)^{-1} f, \varphi \rangle = \int_{\partial U} \frac{f \bar{h}}{\phi - \mu} |dz| \bigg/ \int_{\partial U} \frac{\varphi \bar{h}}{\phi - \mu} |dz| = \int_{\partial U} \frac{f \bar{h}}{z - \lambda} |dz| \bigg/ \int_{\partial U} \frac{\varphi \bar{h}}{z - \lambda} |dz|$$

and the last expression equals the value of the pseudocontinuation of f/φ to G at λ . Thus, it follows as in the proof given above that we need to verify that $\limsup_{r \rightarrow 1} (1 - r^2) \int_{|\mu|=r} \|L_\mu f\|^2 \frac{|d\mu|}{2\pi r} = 0$, where $L_\mu f(z) = (1 - \mu L)^{-1} f = \frac{f(z) - (f/\varphi)(\phi^{-1}(\mu))\varphi(z)}{\phi(z) - \mu}$. The proof of this can be given with the same arguments as above.

Finally, we would like to use this opportunity to point out another omission in the paper: In Lemma 3.1, the pseudocontinuation for f/g is given by $f/g = T_h f / T_h g$ for all functions $h \in \mathcal{M}^\perp$ such that $T_h g$ is not identically equal to zero.

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