Correction to

Group Algebras of Finite Representation Type

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Recently we have observed that Lemma 4 in [8] is not true and hence the characterization of group algebras of finite representation type given in [8] is not complete.

Here, we will give the correct version of [8, Theorem] and we will complete the proof given there.

We use the same notation as in [8]. In order to state the main result, we need some additional. Let F be a division algebra over a fixed field K . We will denote by $R_F(1)$ the quiver algebra FQ_1 (in the sense of [3]) of the quiver

$$Q_1: 1 \rightarrow 2 \leftarrow 3$$

and by $R_F(2)$ the opposite algebra to $R_F(1)$. Moreover, we will denote by $R_F(3)$ the bounden quiver algebra FQ_2/I_2 where Q_2 is the quiver

$$1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4$$

and I_2 is the ideal in the quiver algebra FQ_2 generated by $\alpha_2\alpha_3$, and by $R_F(4)$ the opposite algebra to $R_F(3)$. Finally, recall that for nonnegative integers m and n , $R_F(m, n)$ is the bounden quiver algebra $FQ_{m,n}/I_{m,n}$ where $Q_{m,n}$ is the quiver

$$\begin{array}{ccccccc} \bullet & \xrightarrow{\beta_m} & \bullet & \rightarrow & \dots & \xrightarrow{\beta_1} & \bullet \\ -m & & -m+1 & & & & 0 \\ & & & & & \xrightarrow{\alpha_1} & \bullet \\ & & & & & & 1 \\ & & & & & \rightarrow & \dots \\ & & & & & & n-1 \\ & & & & & \xrightarrow{\alpha_n} & \bullet \\ & & & & & & n \end{array}$$

and $I_{m,n}$ is the ideal in the quiver algebra $FQ_{m,n}$ generated by the composed arrows $\beta_i\beta_{i+1}$, $i=1, \dots, m-1$, and $\alpha_{j+1}\alpha_j$, $j=1, \dots, n-1$.

Theorem. *Let G be a finite group and let R be a connected finite-dimensional basic algebra over a field K of characteristic p . Then RG is of finite representation type if and only if one to the following cases holds.*

- (i) p does not divide the order of G and R is of finite representation type.
- (ii) p divides the order of G and one of the following holds:

- (1) R is a division algebra and a p -Sylow subgroup of G is cyclic or
 (2) $p=3$, a 3-Sylow subgroup of G is simple, and R is isomorphic to an algebra $R_F(0, 1)$ or
 (3) $p=2$, a 2-Sylow subgroup of G is simple, and R is isomorphic to an algebra $R_F(m, n)$, to an algebra $R_F(i)$, $i=1, 2, 3, 4$, or to a triangular matrix algebra $\begin{pmatrix} F & {}_F M_H \\ 0 & H \end{pmatrix}$, where F and H are division algebras, ${}_F M_H$ is an F - H -bimodule, K acts centrally on F , H , and ${}_F M_H$, and $\dim_F({}_F M_H) \cdot \dim({}_F M_H)_H = 2$.

Let R be a non-simple algebra over a fixed field K of characteristic $p > 0$ and let G be a cyclic group of order $n = p^k$. From [8], for our aim, we can assume that R is isomorphic to a bounden quiver algebra FQ/I , where F is a division algebra and Q is of the form

$$Q: 1 \xrightarrow{\alpha_1} 2 \cdots m-1 \xrightarrow{\alpha_{m-1}} m$$

and $I(i, j) = 0$ if i and j are neighbours in Q . Then from [8, Lemma 3] RG is isomorphic to the bounden quiver algebra FQ'/I' where Q' is the quiver

$$Q': 1 \xrightarrow{\beta_1} 2 \xrightarrow{\alpha_2} \cdots m-1 \xrightarrow{\alpha_{m-1}} m$$

and I' is generated by I , β_i^n , $i=1, \dots, m$, $\alpha_i \beta_i - \beta_{i+1} \alpha_i$ if $i \xrightarrow{\alpha_i} i+1$ and $\beta_i \alpha_i - \alpha_i \beta_{i+1}$, if $i \xleftarrow{\alpha_i} i+1$, $i=1, \dots, m-1$. Let us denote by $(\bar{Q}, \bar{I})$ the bounden quiver constructed in the following way: vertices of $\bar{Q}$ are pairs $(i, j) \in \{1, \dots, m\} \times \mathbb{Z}$, $\mathbb{Z}$ is the ring of integers, arrows of $\bar{Q}$ are formed by $(i, \beta_j): (i, j) \rightarrow (i, j-1)$, $i=1, \dots, m$, $(\alpha_i, j): (i, j) \rightarrow (i+1, j)$ if $\alpha_i: i \rightarrow i+1$ in Q and $(\alpha_i, j): (i+1, j) \rightarrow (i, j)$ if $\alpha_i: i+1 \rightarrow i$ in Q , and $\bar{I}$ is generated by elements $(\alpha_k, j) \dots (\alpha_i, j)$ if $\alpha_k \dots \alpha_i$ belongs to I , $(i, \beta_j)(i, \beta_{j+1}) \dots (i, \beta_{j+n-1})$, $(\alpha_i, j-1)(i, \beta_j) - (i+1, \beta_j)(\alpha_i, j)$ if $i \xrightarrow{\alpha_i} i+1$ and $(\alpha_i, j-1)(i+1, \beta_j) - i, \beta_j)(\alpha_i, j)$ if $i \xleftarrow{\alpha_i} i+1$, $i, k \in \{1, \dots, m\}$, $j \in \mathbb{Z}$. Let σ be the automorphism of $(\bar{Q}, \bar{I})$ given by $\sigma((i, j)) = (i, j-1)$, $\sigma((\alpha_i, j)) = (\alpha_i, j-1)$, $\sigma((i, \beta_j)) = (i, \beta_{j-1})$ and let $\langle \sigma \rangle$ be the cyclic group generated by σ .

Then $(\bar{Q}, \bar{I})$ defines a locally finite dimensional K -category, the action of $\langle \sigma \rangle$ on $(\bar{Q}, \bar{I})$ is free and locally bounded, the quotient bounden quiver $(\bar{Q}, \bar{I})/\langle \sigma \rangle$ is isomorphic to (Q', I') , and the canonical projection $(\bar{Q}, \bar{I}) \rightarrow (\bar{Q}, \bar{I})/\langle \sigma \rangle$ is a Galois covering. From [4, 5] we know that (Q', I') is representation-finite if and only if $(\bar{Q}, \bar{I})$ is locally representation-finite.

In order to complete the proof of Theorem given in [8] we must examine carefully the case when Q contains, as a subquiver, one of the following quivers

$$\bullet \rightarrow \bullet \leftarrow \bullet \quad \text{or} \quad \bullet \leftarrow \bullet \rightarrow \bullet$$

By duality it is enough to consider only the first case.

Lemma 1. $R_F(1)$ G is of finite representation type if and only if $n=2$.

Proof. Assume that $n \geq 3$. Then $\text{mod}_F(\bar{Q}_1, \bar{I}_1)$ contains as a full subcategory the category of all finite-dimensional representation of the following extended

Dynkin graph of type $\tilde{D}_4$

Hence $\text{mod}_F(\bar{Q}_1, \bar{I}_1)$ is not locally representation-finite and $R_F(1)G$ is not of finite representation type. Assume $n=2$. In order to prove that $(\bar{Q}_1, \bar{I}_1)$ is locally representation-finite we will prove the following: if $(V_{(i,j)})$ is an indecomposable object in $\text{mod}_F(\bar{Q}_1, \bar{I}_1)$ such that $V_{(i,j)}=0$ for $j<0$ and $V_{(i,0)}\neq 0$ for some i , then $V_{(i,j)}=0$ for $j\geq 3$. Thus our claim concerns the full subquivers (X_t, J_t) of $(\bar{Q}_1, \bar{I}_1)$ formed by the vertices (i, j) , $i=1, 2, 3$, $j=0, \dots, t$, and induced relations. The determination of the representation type of (X_t, J_t) and the computation of its Auslander-Reiten quiver can be done by calculation of dimension vectors (see [3]). The computation is recorded in Fig. 1. The double arrows denote the periodic part of the Auslander-Reiten quiver of (X_3, J_3) . Repeating this periodic part infinitely many times on the left and on the right, we obtain the periodic Auslander-Reiten quiver of $(\bar{Q}_1, \bar{I}_1)$. Consequently $R_F(1)G$ has 42 nonisomorphic indecomposable right modules and this finishes the proof of the lemma.

Lemma 2. *Let $n=2$ and let R be the quiver algebra FQ where Q is one of the quivers*

$$1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \quad \text{or} \quad 1 \rightarrow 2 \leftarrow 3 \leftarrow 4.$$

Then RG is of infinite representation type.

Proof. It is not hard to see that $\text{mod}_F \bar{Q}$ contains as a full subcategory the category $\text{mod}_F X$ where X is the following extended Dynkin graph of type $\tilde{D}_5$

From [1], FX is a hereditary algebra of finite representation type, $\text{mod}_F \bar{Q}$ is not locally representation-finite, and consequently RG is not of finite representation type.

Lemma 3. *Let $n=2$ and let R be the bounden quiver algebra FQ/I where*

$$Q: 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

and I is generated by the composed arrows $\alpha_2 \alpha_3$ and $\alpha_3 \alpha_4$. Then RG is not of finite representation type.

Fig. 1

Proof. It is easy to see that $\text{mod}_F(\bar{Q}, \bar{I})$ contains as a full subcategory the category of all finite-dimensional representations of the following extended Dynkin graph of type $\tilde{E}_8$

$$\begin{array}{c} (2, 3) \\ \downarrow \\ (1, 1) \leftarrow (1, 2) \rightarrow (2, 2) \leftarrow (3, 2) \rightarrow (3, 1) \leftarrow (4, 1) \rightarrow (4, 0) \leftarrow (5, 0). \end{array}$$

Then as above we conclude that RG is not of finite representation type.

Lemma 4. Let $n=2$ and let R be the bounden quiver algebra FQ/I where

$$Q: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xleftarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

and I is generated by the composed arrows $\alpha_2\alpha_1$ and $\alpha_3\alpha_4$. Then RG is not of finite representation type.

Proof. It is not hard to see that $\text{mod}_F(\bar{Q}, \bar{I})$ contains as a full sub-category the category of all finite-dimensional representations of the following extended

Fig. 1 (continuation)

Dynkin graph of type $\tilde{E}_7$

Then as above we conclude that RG is not of finite representation type.

From [8] and the above lemmas, for the proof of Theorem, it is sufficient to prove the following lemma.

Lemma 5. *If $R = R_F(3)$ and $n = 2$, then RG is of finite representation type.*

Proof. From [4, Theorem 3.6] it is enough to show that $(\bar{Q}_2, \bar{I}_2)$ is locally representation-finite. For each $t \geq 0$, let (X_t, J_t) be the full bounden subquiver of $(\bar{Q}_2, \bar{I}_2)$ formed by the vertices (i, j) , $i = 1, 2, 3$, $j \geq 0$, and $(4, j)$, $j = 0, \dots, t$, and induced relations. We will show that each bounden quiver (X_t, J_t) is locally representation-finite. We shall proceed by induction on t using one point extension procedure. From Lemma 1 we know that the full bounden subquiver (X, J) of $(\bar{Q}_2, \bar{I}_2)$ formed by the vertices (i, j) , $i = 1, 2, 3$, $j \geq 0$, is locally repre-

sentation-finite. For each point (i, j) of $(\bar{Q}_2, \bar{I}_2)$, we denote by $S_{(i, j)}$ the simple representation of $(\bar{Q}_2, \bar{I}_2)$ given by (i, j) . Then, in notation of [8], each indecomposable representation of (X_0, J_0) is either an indecomposable representation of (X, J) , is isomorphic to $S_{(4, 0)}$, or is an indecomposable object in the subspace category $\mathcal{U}(\mathcal{N}_F)$ where $N = S_{(3, 0)}$ (see also [7]). From Fig. 1 it easily follows that $\mathcal{U}(\mathcal{N}_F)$ is equivalent to $\mathcal{U}(\mathcal{T}_F)$ where T is the partially ordered set given by the following family of irreducible maps in $\text{mod}_F(X, J)$

$$\begin{array}{ccccccccc}
 000 & 000 & 010 & 010 & 010 & & & & \\
 000 \rightarrow 011 \rightarrow 121 \rightarrow 121 \rightarrow 011 & & & & & & & & \\
 001 & 001 & 101 & 001 & 001 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 000 & 010 & 010 & & & & \\
 & & 111 \rightarrow 232 \rightarrow 122 & & & & & & \\
 & & 101 & 101 & 101 & & & & \\
 & & & \downarrow & \downarrow & & & & \\
 & & & 010 & 020 & 010 & & & \\
 & & & 221 \rightarrow 232 \rightarrow 111 & & & & & \\
 & & & 101 & 101 & 101 & & & \\
 & & & \downarrow & \downarrow & \downarrow & & & \\
 & & & 000 & 010 & 010 & 010 & 110 & 000 \\
 & & & 111 \rightarrow 122 \rightarrow 222 \rightarrow 111 \rightarrow 111 \rightarrow 001 & & & & & \\
 & & & 001 & 001 & 101 & 001 & 001 & 001.
 \end{array}$$

Hence from [6] (see also [8, Lemma 8]). (X_0, J_0) is locally representation-finite. For $t=1$, each indecomposable representation of (X_1, J_1) is either an indecomposable representation of (X_0, J_0) , is isomorphic to $S_{(4, 1)}$, or is an indecomposable object in the subspace category $\mathcal{U}(\mathcal{E}_F)$ where E is the injective envelope of $S_{(3, 0)}$ in $\text{mod}_F(X_0, J_0)$. Observe that $E/S_{(3, 0)}$ is isomorphic to $S_{(3, 1)} \oplus S_{(4, 0)}$, and $S_{(4, 0)}$ is a simple injective object in $\text{mod}_F(X_0, J_0)$. Since there are no nonzero maps from $S_{(3, 1)}$ to indecomposable objects in $\text{mod}_F(X_0, J_0)$ forming the above partially ordered set T , $\mathcal{U}(\mathcal{E}_F)$ is equivalent to $\mathcal{U}(\mathcal{L}_F)$ where L is the partially ordered set whose elements are $S_{4, 0}$, E , and the objects from $\sigma^{-1}(T)$, and with the order: $E < S_{4, 0}$, $E < Y$ for any Y from $\sigma^{-1}(T)$, and $\sigma^{-1}(V) < \sigma^{-1}(W)$ if and only if $V < W$ in T . Then from [2, 6], (X_1, J_1) is locally representation-finite. Observe also $S_{(4, 1)}$ is a simple injective object in $\text{mod}_F(X_1, J_1)$ and there are no nonzero maps from $S_{(3, 2)}$ to indecomposable objects in $\text{mod}_F(X_1, J_1)$ which are not in $\text{mod}_F(X, J)$. Hence repeating the above arguments we conclude that each bounden quiver (X_t, J_t) is locally representation-finite. Moreover, if $M = (M_{(i, j)})$ is an indecomposable representation of $(\bar{Q}_2, \bar{I}_2)$ such that $M_{(i, j)} = 0$ for $j < 0$ and $M_{(i, j)} \neq 0$ for some i , then M is an indecomposable representation of (X_1, J_1) . Consequently $(\bar{Q}_2, \bar{I}_2)$ is locally representation-finite and $R_F(3) G$ is of finite representation type.

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