Addendum to On Tauberian Theorems for the S_α -Method of Summability

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I supply here an addendum to my paper in this Zeitschrift consisting of observations kindly communicated by Dr. B. Kuttner and arranged in four sections. All the sections adopt without change the notation of my paper referred to; and the first section is a necessary addition to that paper.

I

Lemma 6, on p. 41, has been stated for all $\{s_n\}$ bounded on one side but proved only for s_n remaining on one side of its S_α -limit. Thus the lemma stands unproved though its important case required for application in the paper may be restated as below and proved as suggested by Dr. Kuttner.

Lemma 6'. *If $r \geq 0$ is given (not necessarily an integer) and $\{s_n\}$ is bounded, then the hypothesis*

$$S_\alpha(m+r) \rightarrow l \quad \text{as } m \rightarrow \infty$$

implies the conclusion

$$s_n \rightarrow l(S_\alpha). \quad (\text{a})$$

Proof. The conclusion (a) will follow if we assume (clearly without loss of generality) that $r = \vartheta$, $0 < \vartheta < 1$ (ϑ fixed), and show that

$$\sum_{n=0}^{\infty} |c_n(m) - c_n(m+\vartheta)| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (\text{b})$$

By definition

$$\log c_n(x) = (x+1) \log(1-\alpha) + \sum_{v=1}^n \log(x+v) - \log(n!) + n \log \alpha.$$

Hence, using dashes to denote differentiation with respect to x , we see that

$$\frac{c'_n(x)}{c_n(x)} = \log(1-\alpha) + \sum_{v=1}^n \frac{1}{x+v}. \quad (\text{c})$$

Here $\log(1-\alpha)$ is a negative constant. Also, for fixed n and $x > -1$, the sum on the right side of (c) is a positive decreasing function of x which tends to ∞ as $x \rightarrow -1$ and to 0 as $x \rightarrow \infty$. Thus there is an $x_0(n) > -1$ such that the expression on the right side of (c) is positive for $-1 < x < x_0(n)$ and negative for $x > x_0(n)$. Hence $c_n(x)$ is (strictly) increasing for $-1 < x < x_0(n)$, and (strictly) decreasing for $x > x_0(n)$.

By definition again

$$\frac{c_n(x)}{c_n(x-1)} = (1-\alpha) \left(\frac{n+x}{x} \right)$$

whence it follows that

$$c_n \left(\frac{n(1-\alpha)}{n} - 1 \right) = c_n \left(\frac{n(1-\alpha)}{n} \right).$$

Therefore

$$\frac{n(1-\alpha)}{\alpha} - 1 < x_0(n) < \frac{n(1-\alpha)}{\alpha}. \quad (d)$$

And so, if

$$m \leq \frac{n(1-\alpha)}{\alpha} - 2, \quad (e)$$

then, by (d), $m+1 < x_0(n)$, so that

$$c_n(m) < c_n(m+1) < c_n(m+1).$$

On the other hand, if

$$m \geq \frac{n(1-\alpha)}{\alpha}, \quad (f)$$

then $m > x_0(n)$, so that

$$c_n(m) > c_n(m+1) > c_n(m+1).$$

Thus, in either case,

$$|c_n(m) - c_n(m+1)| < |c_n(m) - c_n(m+1)|,$$

except possibly for those values of n for which neither (e) nor (f) holds, i.e. values of n for which

$$\frac{m\alpha}{1-\alpha} < n < \frac{(m+2)\alpha}{1-\alpha}. \quad (g)$$

Now the number of integers n satisfying (g) is clearly a bounded function of m . Also it is well known that

$$\max_n c_n(x) = O(x^{-\frac{1}{2}}) \quad \text{as } x \rightarrow \infty.$$

Hence the contribution to the sum (b) of those terms satisfying (g) is $O(m^{-\frac{1}{2}})$ and thus (b) will follow if we prove that

$$\sum_{n=0}^{\infty} |c_n(m) - c_n(m+1)| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (h)$$

To prove (h) we note that the sum on the left side of (h) is equal to

$$\begin{aligned} & \frac{1}{m+1} \sum_{n=0}^{\infty} c_n(m) |(m+1)\alpha - (1-\alpha)n| \\ & \leq \frac{1}{m+1} \left\{ \sum_{n=0}^{\infty} c_n(m) \right\}^2 \left\{ \sum_{n=0}^{\infty} c_n(m) [(m+1)\alpha - (1-\alpha)n]^2 \right\}^{\frac{1}{2}} \end{aligned}$$

where of course

$$\sum_{n=0}^{\infty} c_n(m) = 1.$$

Also, if in Eq. (13), on p. 38 of my paper cited at the outset,

$$t = (m+1)\alpha/(1-\alpha) = (m+1)\lambda,$$

then

$$\sum_{n=0}^{\infty} c_n(m) [(m+1)\alpha - (1-\alpha)n]^2 = (m+1)\alpha.$$

(h) now follows and thence (b) as also the conclusion (a).

II

The proof of Theorem I', in line 11 from the bottom of p. 46, contains a misprint which should be corrected by reading

$$\sum_{r=m+1}^n \frac{C_r^1 - C_{r-1}^1}{r+1} = \sum_{r=m+1}^n \frac{C_r^0 - C_r^1}{r} \quad \text{for} \quad \sum_{r=m+1}^n \frac{C_r^0 - C_{r-1}^1}{r}.$$

This correction does not affect the rest of the proof which, however, may be presented more simply and clearly by an appeal to Corollary I(1) instead of Theorem I and the explicit use of induction. For the sake of convenience, such a proof of Theorem I', together with a complete statement of Theorem I' itself, is given below.

Theorem I'. *Under the condition*

$$\lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \min_{n \leq m \leq n + \delta \sqrt{n}} \frac{s_m - s_n}{n^k} \geq 0, \quad k \geq 0, \quad (4)$$

or its particular form

$$s_n - s_{n-1} = O_L(n^{k-\frac{1}{2}}), \quad k \geq 0, \quad (4')$$

given in Corollary I(1), together with the condition

$$S_\alpha(m) = O(m^{k-(p-1)/2}) \quad \text{as} \quad m \rightarrow \infty, \quad (29)$$

where p is any one of the integers

$$1, 2, \dots, [2k+1],$$

the last of which is the greatest integer in $2k+1$, we have

$$C_n^{p-1} = O(n^{k-(p-1)/2}) \quad \text{as} \quad n \rightarrow \infty. \quad (30)$$

Proof. If q is an integer such that $2 \leq q \leq p$, then the conclusion (30) holds for $p=q$ whenever it holds for $p=q-1$. To prove this we note that (30) for

$p = q - 1$ gives successively

$$\begin{aligned} C_n^{q-2} &= O(n^{k-(q-2)/2}), & C_n^{q-1} &= O(n^{k-(q-2)/2}), \\ C_n^{q-1} - C_{n-1}^{q-1} &= (C_n^{q-2} - C_{n-1}^{q-1}) n^{-1} = O(n^{k-q/2}). \end{aligned} \quad (4a)$$

On the other hand, hypothesis (29) gives us, as already explained in (31) on p. 46 of my paper

$$\begin{aligned} S_\alpha \{m+q-1; C^{q-1}(\mathfrak{S})\} &= C^{q-1} \{S_\alpha(\mathfrak{S})\} = O\{(m+q-1)^{k-(p-1)/2}\} \\ &= \begin{cases} O\{(m+q-1)^{k-(q-1)/2}\} & \text{if } q=p \\ O\{(m+q-1)^{k-(q-1)/2}\} & \text{if } q < p. \end{cases} \end{aligned} \quad (29a)$$

Now Theorem I, already established in my paper on p. 41, is the particular case of Theorem I' for $p=1$. Appealing to this particular case, with $\{s_n\}$ replaced by $\{C_n^{q-1}\}$ and $k-\frac{1}{2}$ replaced by $k-(q/2)$ in (4') and (29), we conclude from (4'a) and (29a) that

$$C_n^{q-1} = O(n^{k-(q-1)/2}).$$

Thus (30) for $p=q$ follows from (30) for $p=q-1$. Since (30) is true for $p=1$, the proof is now completed by induction.

III

In the footnote on p. 48, it is proved that

$$\frac{E_m^{2k}}{E_n^{2k}} - 1 < \delta O\left(\frac{1}{\sqrt{m}}\right) \quad \text{if } m + \delta\sqrt{m} > n > m \rightarrow \infty.$$

The proof may be simplified as follows.

$$E_n^{2k} - E_m^{2k} = \sum_{v=m+1}^{\infty} E_v^{2k-1}$$

where every term of the sum is asymptotically equal to a constant multiplied by m^{2k-1} . Moreover,

$$1/E_n^{2k} \sim \Gamma(2k+1)/n^{2k} < \Gamma(2k+1)/m^{2k},$$

so that

$$\left| \frac{E_m^{2k}}{E_n^{2k}} - 1 \right| = \frac{E_n^{2k} - E_m^{2k}}{E_n^{2k}} = (n-m) O(m^{2k-1}) O(m^{-2k}) < \delta O(m^{-\frac{1}{2}}).$$

IV

Minor omissions, corrections and explanations to be supplied in my paper are as follows.

The second displayed step on p. 34, the first two displayed steps on p. 35, and the first displayed step in the proof of Theorem II on p. 47 are to be numbered respectively (1₁), (2), (2₁), and (35).

In p. 39, line 9↑, read (8) for (9).

In p. 39, line 4↑, read $(\lambda m)^{k-\frac{1}{2}}$ for λm .

In p. 48, the appeal in the line immediately following (42) is to the case of Lemma 6 proved as Lemma 6' in I above.

In p. (37), the estimate numbered (11₁) tacitly assumes that $n \sim \lambda m$ in the range of summation. This is justified if we suppose (as we may) that $\xi = o(\sqrt{m})$.

In p. (43), the estimate numbered (25) makes the tacit unjustified assumption $s_n \geq -n^k \sigma_2(N)$ for $n=0$. This is avoided if we suppose (as we may) even at the outset that $s_0=0$.

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