Correction to “A Characterization of Some Doubly Transitive Groups”

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I am indebted to Dr. F. Piper for pointing out to me that the proof of Theorem 2 (i) in the above paper is not valid. The theorem should be replaced by Theorem 2 (i)*, stated and proved below. The other results in the paper still remain valid under this alteration.

Theorem 2 (i)*. If G is a transitive collineation group of $PG(m, 2^r)$ such that for each line there is an involution in G that fixes exactly the points of the line, then $m=2$ or 3 . If $m=2$, then $G \supseteq PSL(3, 2^r)$. If $m=3$, then either (1) G is transitive on lines, and $G \supseteq PSL(4, 2^r)$, except when $r=1$ in which case $G = PSL(4, 2)$ or $G \approx A_7$, or (2) G has two orbits of lines, one of length $2^{2r} + 1$.

Proof. The proof that $m=2$ or 3 remains unaltered. If π is any plane in the space, and $G_{(\pi)}$ the subgroup of G fixing π as a set, then $G_{(\pi)}$ induces on π a collineation group that contains for each line L in π , an elation of π with axis L . If $m=2$, then $G_{(\pi)} = G$, and is transitive on points, so that by the dual of Result 2, $G \supseteq PSL(3, 2^r)$. If $m=3$, there are two cases to consider: (a) $G_{(\pi)}$ moves every line of π , or (b) $G_{(\pi)}$ fixes some line in π .

In case (a) we can prove, as in the paper, the stated conclusions of (1) above. In the case (b), which was not considered in the paper, let L be the line in π that is fixed by $G_{(\pi)}$. We call L the special line for the plane π . Since G is transitive on points, it is transitive on planes, so that every plane contains a special line. Transitivity on points also shows that every point lies on a special line, and in fact on exactly one special line, since if L_1 and L_2 were two special lines containing the point α , then the plane containing L_1 and L_2 would have two special lines, which is impossible. This also implies that all the special lines are in one orbit of lines under G , and the orbit must evidently have length $2^{2r} + 1$.

To show that the remaining lines are in one orbit, we need only show that all non-special lines in a plane are in one orbit, since G is transitive on planes. Thus let π be a plane, L its special line, and L_1 and L_2 two distinct non-special lines in π . Let a_i be an involution fixing exactly the points of L_i , for $i=1, 2$. Let L_1 and L_2 intersect at the point α , and suppose that $\alpha \notin L$. Then if $a_1 a_2$ had even order there would be an involution fixing pointwise a line through α with centre α , which is impossible since the centre must lie on L . Thus $a_1 a_2$ has odd order, and some power of it takes L_1 onto L_2 . From this we find that $G_{(\pi)}$ is transitive on non-special lines, and obtain the stated result (2).

That groups satisfying the conclusion (2) do in fact occur, is demonstrated by the following set of examples:

Let V denote the 2-dimensional vector space over $GF(2^{2r})$, $r \geq 1$, and let S denote the group of all semi-linear transformations of V . Then S induces the group $P\Gamma L(2, 2^{2r})$ on the projective line $PG(1, 2^{2r})$. Since V contains 2^{4r} points, we can define it to be a vector space over $GF(2^r)$, which will evidently have dimension 4. If we now form $PG(3, 2^r)$ from this $V_4(2^r)$ in the usual way, we see that S induces a collineation group G on $PG(3, 2^r)$. Points of the projective line $PG(1, 2^{2r})$ now become lines in $PG(3, 2^r)$ that are mapped amongst themselves by G . We call these lines special lines of $PG(3, 2^r)$. No two special lines intersect, and every point of $PG(3, 2^r)$ is on exactly one special line.

We show that G satisfies Theorem 2 (i)* above, and the conclusions of (2).

The order of G is $(2^{2r} + 1) 2^{2r} (2^{2r} - 1) 2r(2^r + 1)$. G is triply transitive on special lines, and the subgroup fixing all the special lines has order $2^r + 1$, and is cyclic on the points of each of these lines. Thus G is transitive on points of $PG(3, 2^r)$.

If L is a special line, then there is an involution in G fixing only the one special line L . Since L consists of $2^r + 1$ points, such an involution, t , must fix at least one point of L , and since t moves all other special lines, t cannot fix any point not on L . As t fixes at least one plane containing L , t must fix exactly the points of L , by a theorem due to Baer on involutions of planes (see Wagner [6], Theorem 4). Since G is transitive on special lines, such involutions exist for each special line.

Let π be any plane in $PG(3, 2^r)$. Since G is transitive on points, it is also transitive on planes, and each plane contains exactly one special line. Let L be the special line in π , and let L^* be any other line in π . Let α and β be two points on L^* not on L , and let L_1 and L_2 be the special lines through α and β respectively. Let t be an involution in G fixing the lines L , L_1 and L_2 . Since the stabilizer in G of the set of special lines is cyclic on the points of each special line, we can choose t to fix the point α of L_1 . Since t then fixes π and L_2 , it must also fix β , and hence L^* . It is easy to see, by consideration of $P\Gamma L(2, 2^{2r})$, that t fixes exactly $2^r + 1$ special lines, and hence at most $2^r + 1$ points of π . By Baer's result, t must fix exactly the $2^r + 1$ points of the line L^* . It follows that for any line in the space there is an involution in G fixing exactly the points of that line. G is clearly not transitive on lines, and thus satisfies (2) in the theorem.

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