implies that the universal covering is a cylinder. Therefore the investigated surface is also a cylinder.

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THEORY OF THE CURVATURE OF FRAMED DISTRIBUTIONS Δ_2 IN E_4

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UDC 513

1. Basic Information

In a four-dimensional Euclidean space E_4 with a curvilinear system of coordinates $\{X^i\}$ we consider the moving frame $R(A, e_i)$, $i = \overline{1, 4}$. The derivational formulas of frame R have the form

$$dA = \omega^i e_i, \quad de_i = \omega_i^k e_k, \quad (1.1)$$

where the linear differential Pfaff forms ω^i , ω_i^k satisfy the equations of a Euclidean space structure $D\omega^i = \omega^j \wedge \omega_j^i$, $D\omega_i^k = \omega_i^j \wedge \omega_j^k$. Here and further on the indices take on the following values: $i, j, k = \overline{1, 2, 3, 4}$; $a, b, \dots = \overline{1, 2}$; $\alpha, \beta, \dots = \overline{3, 4}$. Summation is carried out on the same indices in different equations in the corresponding limits.

At each point A of the space E_4 we specify, according to a specific rule, a 2-plane $T(A)$ passing through the point A . Vectors e_1, e_2 of the frame R are placed in this plane, vectors e_3, e_4 in the orthogonal to it 2-plane $N(A)$. Thereby in a neighborhood of point A a two-dimensional distribution Δ_2 is given. We now specify at each point A of the space E_4 , according to a specific rule, an orthogonal vector p in the normal plane $N(A)$. Such a distribution will be called a framed distribution Δ_2 in E_4 . Curves touching at their every point the 2-plane $T(A)$, which forms with this point an element of distribution Δ_2 , are called the integral curves of the distribution.

Choosing forms ω^i for bases, we can represent forms ω_a^α as their linear combinations:

$$\omega_a^\alpha = \Gamma_{ai}^\alpha \omega^i. \quad (1.2)$$

To displace point A in the 2-plane $T(A)$, conditions

$$\omega_\alpha = 0 \quad (1.3)$$

are fulfilled.

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The system of differential equations (1.3) determines a collection of integral curves of the distribution.

The system of differential equations (1.3) is totally integrable if and only if the conditions $D\omega^\alpha = \omega^i \wedge \omega_i^\alpha = 0$ are fulfilled, whence, using the equation of the Euclidean space structure and equalities (1.2) and (1.3), we obtain

$$G^\alpha = \Gamma_{21}^\alpha - \Gamma_{12}^\alpha = 0. \quad (1.4)$$

When these conditions are fulfilled, the set of the integral curves of distribution Δ_2 , corresponding to the chosen point A of the space E^4 , is located in a neighborhood of this point in a two-dimensional plane of this space.

Taking into account the orthonormality of frame R, we obtain the equality

$$\omega_i^k = -\omega_k^i. \quad (1.5)$$

2. Relative Curvature of the Integral Curve. The Indicatrix

1. We consider an arbitrary integral curve C with tangent vector $\tau = \frac{\omega^a}{ds} e_a = v^a e_a$, where s is the natural parameter of curve C. The relative curvature k_p of the curve C at point A is defined to be the projection of vector $d\tau/ds$ onto vector p .

After differentiating the first of the equations of (1.1) and taking into account the remaining equations of (1.1) and equality (1.3), we obtain

$$d^2 A = d\omega^i e_i + \omega^i \omega_i^k e_k. \quad (2.1)$$

The formula

$$k_p = \frac{\omega^a}{ds} \cdot \frac{\omega_a^\alpha}{ds} p_\alpha, \quad (2.2)$$

follows from the definition of k_p if, according to equalities (1.2) and (1.3),

$$k_p = \Gamma_{ab}^\alpha v^a v^b p_\alpha. \quad (2.2')$$

Remark 1. Since the expression of k_p contains only the values $\Gamma_{ab}^\alpha, p_\alpha$ and $\omega^1: \omega^2$ determining the direction of the tangent for the selected curve C, all the curves having a common tangent at point A also have the same relative curvature at this point. This is an analog of Meusnier's theorem.

The magnitude $\rho = 1/|k_p|$, by analogy with the concept of the radius of curvature of a curve on a surface, shall be called the radius of relative curvature of the integral curve C at point A:

$$|k_p| = \frac{1}{\rho} = |\Gamma_{ab}^\alpha v^a v^b p_\alpha|. \quad (2.3)$$

2. We can arrive at the concept of relative curvature differently. Suppose that under a displacement along some direction dA in the 2-plane $T(A)$ we obtain a new distribution plane $T(A') = T(A + dA)$. Vectors p and $p + dp$ will be the vectors of the given vector field at points A and $A + dA$, respectively.

For the measure of the relative curvature of an integral curve C at point A determined by the direction $\tau = dA/ds$, we take the limit, as $A \rightarrow A'$, of the value $(\widehat{p, p + dp})/|AA'|$, that is,

$$k_p = \lim_{A \rightarrow A'} \frac{\sin(\widehat{p, p + dp})}{|AA'|} = \lim_{A \rightarrow A'} \left| \frac{\cos(\tau, p + dp)}{ds} \right| = \lim_{A \rightarrow A'} \frac{|dA \cdot (p + dp)|}{ds^2 |p + dp|} = \lim_{A \rightarrow A'} \frac{|\omega^a e_a (p_\alpha e_\alpha + p_\alpha de_\alpha + dp_\alpha e_\alpha)|}{ds^2 |p + dp|}.$$

Taking into account only the leading terms and the equalities (1.1), we obtain

$$|k_p| = \frac{1}{\rho} = \frac{\sum_\alpha (\omega_\alpha^1 \omega^1 + \omega_\alpha^2 \omega^2) p_\alpha}{ds^2}.$$

According to (1.5) and (1.2), the last equality can be written so:

$$k_p = \frac{1}{\rho} = \frac{|-\omega_a^\alpha \omega^a p_\alpha|}{ds^2} = |\Gamma_{ab}^\alpha v^a v^b p_\alpha|,$$

which coincides with (2.3).

Remark 2. If we consider an integral curve whose osculating plane is determined by direction τ and vector p , then (2.3) determines the principal curvature of this integral curve, that is, the following holds:

THEOREM. The relative curvature of the integral curve C determined by direction τ coincides with the principal curvature of an integral curve having a plane osculating to the 2-plane and spanned by τ and the vector p .

3. The relative curvature of the integral curve C is related to the normal curvature k_n of this curve [1] in the following way. Consider in the normal 2-plane $N(A)$ a unit vector q orthogonal to the vector p . It is not hard to verify that the relation

$$k_p \cdot p + k_q \cdot q = k_n \quad (2.4)$$

holds, where by k_n we denote the normal curvature vector of the integral curve C .

4. In the tangent 2-plane $T(A)$ of distribution Δ_2 in the direction $\tau = v^a e_a$, we lay out a length equal to $\sqrt{\rho}$ and defined by this direction in (2.3). Varying direction τ in a continuous way, we obtain some curve in the plane $T(A)$. Let us find its equation.

Let $x^a = v^a \sqrt{\rho}$ be the coordinates of a point of this curve. Then (2.3) implies that

$$\Gamma_{ab}^\alpha x^a x^b p_\alpha = \pm 1. \quad (2.5)$$

The obtained curve provides an opportunity for tracking the change in the radius of the relative curvature as the direction changes. It is called the indicatrix of the relative curvature of distribution Δ_2 at point A . This curve is of second order. Its type depends on the sign of the expression

$$D = [(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha]^2 - 4\Gamma_{11}^\alpha p_\alpha \cdot \Gamma_{22}^\alpha p_\alpha. \quad (2.6)$$

We can classify the points of the space according to the type of this curve. Specifically, if $D < 0$, $=0$, or >0 , then curve (2.5) will be, respectively, an ellipse, a parabola, or a hyperbola and the point of the space will be called relative elliptic, relative parabolic, or relative hyperbolic, respectively.

3. Relative Principal Curvatures. Total Relative Curvature and Its

Connection to the Total Curvature of Distribution. The Euler

Theorem. Umbilics

1. As in the case of a surface, we call the greatest and least values of ρ the principal radii of relative curvature, the corresponding distributions the relative principal directions, and the corresponding curvatures the relative principal curvatures.

We find their values. For this we shall look for the extremum of the expression $1/\rho = \Gamma_{ab}^\alpha v^a v^b p_\alpha$ under the condition $v^{1^2} + v^{2^2} = 1$. Using the method of Lagrange multipliers, we consider the function $F = 1/\rho - \lambda(v^{1^2} + v^{2^2} - 1)$. Differentiating it and equating to zero the partial derivatives, we obtain

$$\begin{aligned} 2\Gamma_{11}^\alpha p_\alpha v^1 + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha v^2 - 2\lambda v^1 &= 0, \\ (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha v^1 + 2\Gamma_{22}^\alpha p_\alpha v^2 - 2\lambda v^2 &= 0, \\ v^{1^2} + v^{2^2} &= 1. \end{aligned} \quad (3.1)$$

The consistency condition of the first two equations with respect to v^1 and v^2 has the form

$$\begin{vmatrix} 2(\Gamma_{11}^\alpha p_\alpha - \lambda) & (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha \\ (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha & 2(\Gamma_{22}^\alpha p_\alpha - \lambda) \end{vmatrix} = 0,$$

or

$$4\lambda^2 - 4\lambda(\Gamma_{22}^\alpha p_\alpha + \Gamma_{11}^\alpha p_\alpha) + 4\Gamma_{11}^\alpha p_\alpha \Gamma_{22}^\alpha p_\alpha - ((\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha)^2 = 0. \quad (3.2)$$

If the first equations in (3.1) are multiplied by v^1 and v^2 , respectively, and added, then with the help of the last equation of this system we obtain $\Gamma_{11}^\alpha p_\alpha (v^1)^2 + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha v^1 v^2 + \Gamma_{22}^\alpha p_\alpha (v^2)^2 - \lambda((v^1)^2 + (v^2)^2) = 0$, or $1/\rho - \lambda = 0$, that is, Eq. (3.2) gives an expression for the principal radii of relative curvature $1/\rho_1$, $1/\rho_2$.

2. The product of the principal relative curvatures is called the total relative curvature of distribution Δ_2 at a point A and is denoted by $K_{\Pi p}$. The expression for $K_{\Pi p}$, according to Viéte's theorem, takes the form

$$K_{\pi p} = \frac{1}{\rho_1} \cdot \frac{1}{\rho_2} = \lambda_1 \cdot \lambda_2 = \frac{1}{4} (4\Gamma_{11}^\alpha p_\alpha \Gamma_{22}^\alpha p_\alpha - (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha^2). \quad (3.3)$$

If $D > 0$, then $K_{\pi p} < 0$, when $D < 0$, $K_{\pi p} > 0$, and $K_{\pi p} = 0$ for $D = 0$.

Comparing the last expressions with (2.6) we arrive at the following conclusion: at a relatively elliptic point of space the relative total curvature of the distribution is positive, at a relatively hyperbolic point it is negative, and it is equal to 0 only at a relatively parabolic point.

The relative total curvature of distribution is related to the total curvature [see 1], in the following way. On a normal plane $N(A)$ of distribution Δ_2 we consider an orthonormal pair of vectors p and q . Analogously to Subsec. 3 of Sec. 2, we can verify that

$$K_{\pi p} + K_{\pi q} = K_{\pi} \quad (3.4)$$

where the total curvature of distribution Δ_2 at a point A is denoted by K_{π} .

3. The Euler theorem assertion that the sum of the radii of curvature for two mutually orthogonal directions is constant holds in surface theory.

We will show that this theorem also holds in the theory of distributions Δ_2 in E_4 for the radii of relative curvature.

It is clear that the relative principal directions are orthogonal. Indeed, excluding λ from the first two equations of (3.1) we obtain

$$-(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha (v^1)^2 + 2(\Gamma_{11}^\alpha - \Gamma_{22}^\alpha) p_\alpha v^1 v^2 + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha (v^2)^2 = 0. \quad (3.5)$$

This equality implies the relation $(v^1/v^2)_1 \cdot (v^1/v^2)_2 = -1$ which proves the orthogonality of these directions.

Since $\tau(v^1, v^2)$ is a unit vector, it can be represented in the form $\tau(\cos \gamma, \sin \gamma)$. The expression of the relative curvature for this direction takes the form

$$1/\rho_\gamma = k_p = \Gamma_{11}^\alpha p_\alpha \cos^2 \gamma + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha \sin \gamma \cos \gamma + \Gamma_{22}^\alpha p_\alpha \sin^2 \gamma \quad (3.6)$$

Taking the relative principal directions for the directions of vectors e_1 and e_2 , we obtain the condition $(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha = 0$, and then (3.6) takes the form $1/\rho = \Gamma_{11}^\alpha p_\alpha \cos^2 \gamma + \Gamma_{22}^\alpha p_\alpha \sin^2 \gamma$. For the case when $\gamma = 0$, we have $1/\rho_1 = \Gamma_{11}^\alpha p_\alpha$, and we have $1/\rho_2 = \Gamma_{22}^\alpha p_\alpha$ for $\gamma = \pi/2$. These are the relative principal curvatures. Hence, for an arbitrary angle γ we have $1/\rho = \cos^2 \gamma / \rho_1 + \sin^2 \gamma / \rho_2$.

For an orthogonal direction with angle $\gamma' = \gamma + \pi/2$, we get $1/\rho = \sin^2 \gamma / \rho_1 + \cos^2 \gamma / \rho_2$, whence $1/\rho_\gamma + 1/\rho_{\gamma'} = 1/\rho_1 + 1/\rho_2 = \text{const}$, which proves the theorem.

COROLLARY. The equation of the indicatrix of relative curvature (2.5) implies that the relative principal directions correspond to the vertices of the indicatrix. Indeed, if the relative principal directions are taken for the directions of the coordinate axes, then the equation of the indicatrix of relative curvature takes the form $\Gamma_{11}^\alpha p_\alpha (x^1)^2 + \Gamma_{22}^\alpha p_\alpha (x^2)^2 = \pm 1$, and the vertices are intercept points with the coordinate axes corresponding to the relative principal directions.

4. Consider a point at which the radius of relative curvature with respect to all the directions is identical. At this point the coefficients of $(v^1)^2$, $v^1 v^2$, $(v^2)^2$ in the expression for $1/\rho$ must be proportional. At the sought for point the relative curvature will be identical with respect to all the directions on the tangent plane of distribution Δ_2 , but it may be different for different points of the space having given properties.

Such points will be called relative umbilical points. We verify that they form a line in the space E_4 . Its equation is obtained from (3.5) by dividing by $(v^1)^2 + (v^2)^2 = 1$ and by equation relations of the corresponding coefficients. We get

$$\frac{\Gamma_{11}^\alpha p_\alpha}{1} = \frac{(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha}{0} = \frac{\Gamma_{22}^\alpha p_\alpha}{1}.$$

If the coordinate axes are directed along the relative principal directions, then the equation of this line will be $(x^1)^2 + (x^2)^2 = \pm 1$.

Remark. Since the coordinates of vector p do not occur in the equation of the relative umbilical curve, this curve will be umbilical for any choice of the vector p . Therefore, it should be simply called the umbilical line of distribution Δ_2 in E_4 .

4. Mean Relative Curvature of Distribution, Its Connection to the

Mean Curvature; Geometrical Meaning

1. By analogy to surface theory, we call the half-sum of relative principal curvatures the relative mean curvature of distribution $H_p = \frac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$.

From Eq. (3.2) for the determination of relative principal curvature, according to Viète's theorem and bearing in mind that $1/\rho = \lambda$, we obtain

$$H_p = \frac{1}{2} (\Gamma_{11}^\alpha + \Gamma_{22}^\alpha) p_\alpha. \quad (4.1)$$

We can arrive at the concept of relative mean curvature differently. In the tangent plane $T(A)$ choose any orthogonal pair of directions $\tau(v^1, v^2)$ and $t(w^1, w^2)$. From the orthogonality condition of vectors τ and t , we obtain the relation $v^{1^2} + v^{2^2} = 1$, $w^{1^2} + w^{2^2} = 1$, $v^1 w^1 + v^2 w^2 = 0$, whence $v^{1^2} = w^{2^2}$, $v^{2^2} = w^{1^2}$.

For any pair of integral lines of distribution tangent at point A to the selected directions τ and t , we consider the sum of relative curvatures $k_{p_1} + k_{p_2} = \Gamma_{11}^\alpha p_\alpha (v^{1^2} + w^{1^2}) + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha (v^1 w^2 + v^2 w^1) + \Gamma_{22}^\alpha p_\alpha (v^{2^2} + w^{2^2})$. The orthogonality conditions of vectors τ and t imply that $v^1 w^2 + v^2 w^1 = 0$, then $k_{p_1} + k_{p_2} = (\Gamma_{11}^\alpha + \Gamma_{22}^\alpha) p_\alpha$. Comparing the last expression with (4.1), we get

$$H_p = \frac{1}{2} (k_{p_1} + k_{p_2}). \quad (4.2)$$

Thus, we can define the relative mean curvature of a distribution as the half-sum of the relative curvatures of any orthogonal pair of directions of the tangent plane of distribution.

The relative mean curvature of a distribution is related to the mean curvature vector of this distribution [see 2] by the relation

$$H = H_p p + H_q q \quad (4.3)$$

where by H the mean curvature vector of distribution Δ_2 in E_4 at point A , and p and q are an orthonormal pair of vectors on the plane $N(A)$.

2. We will show that the relative mean curvature of a distribution generalizes the properties of the mean curvature of a surface; and specifically, the vanishing of the relative mean curvature at every point of some domain of the space characterizes the relative minimality of the distribution.

Consider in the tangent plane $T(A)$ of a distribution the arbitrary vector $dA = \omega^a e_a$.

As was introduced in [2], the first quadratic form of an integral curve is defined to be $ds^2 = (\omega^1)^2 + (\omega^2)^2$. By analogy with a surface, we denote the coefficients of $(w^1)^2$, $w^1 w^2/2$, $(w^2)^2$ by E , F , and G , respectively. Then $ds^2 = Ew^{1^2} + 2Fw^1 w^2 + Gw^{2^2}$.

In this case $E = G = 1$, $F = 0$.

We vary every point of the initial integral curve along the vector λp , where $\lambda = \lambda(s)$; we get $R = r + \lambda p$, where r is the vector radius of the initial curve and R of the varied curve.

We denote by $d\tilde{s}^2$ the first quadratic form of the obtained curve; $d\tilde{s}^2 = (dr + d\lambda p + \lambda dp)^2$. Set $\lambda = \sigma\delta$, where δ is an infinitesimal quantity, then $d\tilde{s}^2 = ds^2 + 2\delta\sigma(dr p_\alpha de_\alpha) + [\delta^2]$.

Using the expansion of (1.1) and the representation of vector p , we get $dr dp = (\omega_\alpha^1 \omega^1 + \omega_\alpha^2 \omega^2) p_\alpha$.

Denoting $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ for $d\tilde{s}^2$, analogously to E , F , and G for ds^2 , and taking into account only the leading terms, we obtain

$$\tilde{E} = E - 2\delta\sigma\Gamma_{11}^\alpha p_\alpha, \quad \tilde{F} = F - \delta\sigma(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha, \quad \tilde{G} = G - 2\delta\sigma\Gamma_{22}^\alpha p_\alpha.$$

If we choose for the curve C under consideration a geodesic "straightest" line [see 3], determined by point A and the selected direction $\omega^1: \omega^2$, then the magnitude $\tilde{E}\tilde{G} - \tilde{F}^2$ will determine an element of the osculating geodesic surface and $\tilde{E}\tilde{G} - \tilde{F}^2$ an element of the varied surface. Taking into account only leading terms, we obtain

$$\tilde{E}\tilde{G} - \tilde{F}^2 = EG - F^2 - 2E\delta\sigma\Gamma_{22}^\alpha p_\alpha - 2G\delta\sigma\Gamma_{11}^\alpha p_\alpha + 2F\delta\sigma(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha) p_\alpha.$$

We integrate this equality with respect to area elements of the osculating geodesic surface. For the initial surface to have a smaller area in comparison to the varied one, the relative mean curvature must be equal to zero, that is, $(\Gamma_{11}^\alpha + \Gamma_{22}^\alpha)p_\alpha = 0$.

The converse is also true: when the relative mean curvature vanishes, the area of the osculating geodesic surface is the least under variation along vector p .

Remark 1. Since in our discussion we do not go beyond the limits of the second differential neighborhood, by virtue of the Bompiani theorem for distribution Δ_2 [see 4], the proven fact remains true for any osculating surface of the distribution.

Remark 2. By virtue of relation (4.3) we can conclude that a distribution minimal with respect to the two orthogonal vectors p and q is minimal.

5. First and Second Genus Lines of Relative Curvature; the Connection between Them. Relative Gaussian Curvature; Its Connection to the Gaussian Curvature of a Distribution

In surface theory, curvature lines are obtained by proceeding from two definitions leading to identical results:

- a) curves lying on a surface and having as their tangents the directions of the axes of the normal curvature indicatrix, or, which is the same, the directions of the tangent principal normal sections;
- b) lines at the infinitely close points of which the normals to the surface intersect.

As is known [see 5], the carrying over of these constructions to systems of integral curves of distribution Δ_2 in E_3 leads to various systems of curves.

In this section we solve the problem of constructing curvature lines of a two-dimensional distribution in a four-dimensional Euclidean space.

1. First genus lines of relative curvature are defined to be the integral lines of distribution enveloped by the directions of external relative curvatures.

The equations of these lines can be obtained from Eq. (3.1) for the determination of the relative principal curvatures. Excluding λ from the first two equations of this system, we obtain equality (3.5) and after multiplying it by ds^2 , the equation of the first genus line of relative curvature takes the form

$$-(\Gamma_{12}^\alpha + \Gamma_{21}^\alpha)p_\alpha\omega^1 + 2(\Gamma_{11}^\alpha - \Gamma_{22}^\alpha)p_\alpha\omega^1\omega^2 + (\Gamma_{12}^\alpha + \Gamma_{21}^\alpha)p_\alpha\omega^2 = 0. \quad (5.1)$$

This quadratic equation defines two systems of integral curves. Thus, through each point of the space two lines pass such that at the point A they correspond to tangents to lines having the greatest and least radii of relative curvature.

Returning to the corollary of Subsec. 3 of Sec. 3, we conclude that at each point of the space, the first genus lines of relative curvature are directed along the axes of the relative curvature indicatrix of the distribution. Consequently, at each point of the space, these lines form an orthogonal pair of integral lines.

2. Second genus lines of relative curvature are defined to be the integral lines distribution along which the framed normals, vectors p , intersect at infinitely close points.

We find the equation of these lines. According to the definition, there exist values γ and λ such that relation

$$dA = \gamma p + \lambda(p + dp) \quad (5.2)$$

holds.

Differentiating the equality $p = p_\alpha e_\alpha$, we get $dp = dp_\alpha e_\alpha + p_\alpha de_\alpha$, while taking into account equality (1.1) we have $dp = dp_\alpha e_\alpha + p_\alpha \omega_\alpha^k e_k$. Writing out coordinate-wise the initial equality and taking into account (1.3) and (1.1), we find $\omega^a = \gamma \cdot 0 + \lambda p_\alpha \omega_\alpha^a$, $0 = \gamma \cdot p_\alpha + \lambda(p_\alpha + dp_\alpha + p^\beta \omega_\beta^\alpha)$.

From the last two equalities, taking into account only the leading terms, we get $\lambda = -\gamma$.

Excluding γ from these equations, we get

$$\begin{vmatrix} \omega^1 & \omega_1^\alpha p_\alpha \\ \omega^2 & \omega_2^\alpha p_\alpha \end{vmatrix} = 0,$$

and the equation of second genus lines of relative curvature, using expansions (1.2), takes the form

$$\Gamma_{21}^\alpha p_\alpha \omega^{1*} + (\Gamma_{22}^\alpha - \Gamma_{11}^\alpha) p_\alpha \omega^1 \omega^2 - \Gamma_{12}^\alpha p_\alpha \omega^{2*} = 0. \quad (5.3)$$

3. a) Comparing Eqs. (5.1) and (5.3), which determine first and second genus lines of relative curvature, we see that these equations coincide when the integrability conditions (1.4) of a distribution are satisfied. We will show that they coincide only when the integrability conditions are satisfied.

Consider the difference of the left-hand sides of Eqs. (5.1) and (5.3), multiplying beforehand the latter equation by two. We obtain $G^\alpha p_\alpha (\omega^{1*} + \omega^{2*}) = G^\alpha p_\alpha ds^2 = 0$.

Since vectors e_3 and e_4 of frame R can be chosen arbitrarily in the plane $N(A)$ and, consequently, the coefficients of the vector can be anything, the last expression vanishes only when conditions (1.4) are satisfied, that is, only in the case of the total integrability of distributions Δ_2 .

Thus, two systems of lines of relative curvature coincide only when the integrability conditions are satisfied, that is, in the case of a surface. Otherwise, these are two different systems of curves.

b) Equation (5.3) implies that in the general case, the lines of relative curvature will not be orthogonal, since by Viète's theorem, we can write for Eq. (5.3) $(\omega^1/\omega^2)_1 \cdot (\omega^1/\omega^2)_2 = -\Gamma_{12}^\alpha p_\alpha / \Gamma_{21}^\alpha p_\alpha \neq -1$.

It will be orthogonal only in the case of the total integrability of the distribution, that is, when this system coincides with the system of first genus lines of relative curvature.

c) If the directions of the bisectrices of the first genus lines of relative curvature are taken to be the directions of the coordinate axes, then we have $(\Gamma_{11}^\alpha - \Gamma_{22}^\alpha) p_\alpha = 0$. Here for the directions of second genus lines of relative curvature we have $(\omega_1/\omega_2)_{1,2} = \pm \sqrt{\Gamma_{12}^\alpha p_\alpha / \Gamma_{21}^\alpha p_\alpha}$, that is, these directions are symmetrical with respect to the coordinate axes. Consequently, the directions of the first and second genus lines of relative curvature have common bisectrices.

4. a) We calculate the product of relative curvatures for second genus lines of relative curvature.

For this we present expression (2.2) of the relative curvature in the form

$$k_p = \frac{\omega_1^\alpha \omega^1 + \omega_2^\alpha \omega^2}{\omega^{1*} + \omega^{2*}} p_\alpha. \quad (5.4)$$

Equalities (5.2) are multiplied by ω^1 and ω^2 , respectively, and added, then $\omega^{1*} + \omega^{2*} = \gamma(\omega_1^\alpha \omega^1 + \omega_2^\alpha \omega^2) p_\alpha$, whence $k_p = 1/\gamma$. To find γ we write the consistency condition of (5.2) with respect to ω^1 and ω^2 and get

$$\begin{vmatrix} 1 - \gamma \Gamma_{11}^\alpha p_\alpha & -\gamma \Gamma_{12}^\alpha p_\alpha \\ -\gamma \Gamma_{21}^\alpha p_\alpha & 1 - \gamma \Gamma_{22}^\alpha p_\alpha \end{vmatrix} = 0,$$

whence $\gamma^2 (\Gamma_{11}^\alpha p_\alpha \Gamma_{22}^\beta p_\beta - \Gamma_{12}^\alpha p_\alpha \Gamma_{21}^\beta p_\beta) - \gamma (\Gamma_{11}^\alpha p_\alpha + \Gamma_{22}^\alpha p_\alpha) + 1 = 0$. Solving this equality we obtain γ_1 and γ_2 , corresponding to second genus lines of relative curvature. The product of the corresponding relative curvatures is $k_{p_1} \cdot k_{p_2} = \frac{1}{\gamma_1} \cdot \frac{1}{\gamma_2}$, and, according to Viète's theorem,

$$k_{p_1} k_{p_2} = \Gamma_{11}^\alpha p_\alpha \Gamma_{22}^\beta p_\beta - \Gamma_{12}^\alpha p_\alpha \Gamma_{21}^\beta p_\beta. \quad (5.5)$$

We call this value the relative Gaussian curvature of distribution Δ at a point A and denote it by $K_{\Gamma p}$.

b) Comparing expression (5.5) for $K_{\Gamma p}$ to expression (3.3) for the relative total curvature $K_{\Pi p}$ of distribution Δ , we arrive at the conclusion that they are related by the relation

$$K_{\Gamma p} = K_{\Pi p} - \frac{1}{4} (G^\alpha p_\alpha)^2. \quad (5.6)$$

Consequently, the relative total and Gaussian curvatures coincide only when the integrability conditions are satisfied.

c) The relative Gaussian curvature of a distribution is related to the Gaussian curvature [1] in the following way:

$$K_r = K_{r_p} + K_{r_q} \quad (5.7)$$

where we denote by K_r the Gaussian curvature of distribution Δ at a point A , and p and q are an orthonormal pair of vectors of the plane $N(A)$.

Remark. The half-sum of the relative curvatures for second genus lines of relative curvature, according to Viète's theorem, gives $\frac{1}{2}(\gamma_1 + \gamma_2) = \frac{1}{2}(\Gamma_{11}^\alpha \rho_\alpha + \Gamma_{22}^\alpha \rho_\alpha) = H_p$ - the mean relative curvature (4.1).

5. We can define the relative Gaussian curvature, by analogy to surface theory, through a spherical mapping. Specifically in the tangent plane $T(A)$, we choose arbitrarily two directions $dA = \omega^a e_a, \bar{d}A = \bar{\omega}^a e_a$. We consider the area of triangle $\Delta AA'A''$, where $A' = A + dA, A'' = A + \bar{d}A$. This area $S_{\Delta AA'A''}$ is expressed through vectors dA and $\bar{d}A$ in the following way:

$$S = \sqrt{|dA|^2 |\bar{d}A|^2 - (dA \bar{d}A)^2} / 2 = |\omega^1 \bar{\omega}^2 - \bar{\omega}^1 \omega^2| / 2.$$

At the points $A, A',$ and A'' we consider vectors $p, p + dp, p + \bar{d}p$, respectively. Taking these vectors to point A we obtain a spherical triangle. Since its sides are small, they can be considered segments of straight lines. We project the obtained triangle onto the plane $T(A)$ and compute its area $S^1 = |p_\alpha \omega_\alpha^1 p_\beta \bar{\omega}_\beta^2 - p_\alpha \bar{\omega}_\alpha^1 p_\beta \omega_\beta^2| / 2$. Taking into account equalities (1.2) and (1.5), we determine $S^1 = |\Gamma_{11}^\alpha \rho_\alpha \Gamma_{22}^\beta \rho_\beta - \Gamma_{12}^\alpha \rho_\alpha \Gamma_{21}^\beta \rho_\beta| S$. Considering the ratio of the areas of the obtained and the original triangles, we find a value independent of the choice of initial direction. We can give this value a sign, assuming that the mixed vector product $\Pi p_{T(A)} \bar{d}p, \Pi p_{T(A)} d\bar{p}, e_3, e_4$ occurs in the numerator. Then the last value takes on the form $\Gamma_{11}^\alpha \rho_\alpha \Gamma_{22}^\beta \rho_\beta - \Gamma_{12}^\alpha \rho_\alpha \Gamma_{21}^\beta \rho_\beta$, which coincides with the expression of K_{r_p} .

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