Addendum to "A Priori Estimates for Solutions of the Minimal Surface Equation"

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The purpose of this note is to give an improved version of a basic a priori estimate for the gradient of solutions of the minimal surface equation. Specifically, we intend to prove the following result.

Theorem. *Let $u=u(x, y)$ be a non-negative solution of the minimal surface equation*

$$(1) \quad (1+q^2)t - 2pq s + (1+p^2)t = 0$$

in the unit disk $x^2 + y^2 < 1$. Then

$$(2) \quad W(0) \leq e^{\frac{1}{2}\pi m} \quad (m = u(0)).$$

Here $u(0)$ and $W(0)$ denote respectively the value of u and of $W = \sqrt{1+p^2+q^2}$ at the origin.

In our earlier paper [1] we have given a similar though less sharp result, the conclusion there being

$$W(0) \leq e^{\frac{\pi}{2}m} + m e^{-\frac{\pi}{2}m}.$$

The new version is thus simpler and clearer, and in addition is best possible in the sense that neither the coefficient $\frac{1}{2}\pi$, nor the factor 1 before $e^{\frac{1}{2}\pi m}$, can be made smaller without affecting the validity of the conclusion. Although we shall not give any applications here, it is well known that gradient estimates similar to (2) are of fundamental importance in the study of boundary value problems. It is this fact which justifies the effort involved in obtaining the best possible form for this estimate (we note also that the above result solves the sixteenth problem in the list of open questions in NITSCHÉ's survey [2] of the theory of minimal surfaces).

The proof follows the same outline as before, relying on a basic method due to FINN together with the construction of an appropriate comparison function. For brevity we refer the reader to the earlier paper for those portions of the argument which carry over unchanged. We emphasize, however, that the details remain entirely elementary.

1.

It will be sufficient to consider a *positive* solution $u=u(x, y)$ of the minimal surface equation in the *closed* unit disk. Moreover, for the purpose of demonstrating (2) it can be assumed without loss of generality that

$$(3) \quad \frac{\partial u}{\partial x} > 0, \quad \frac{\partial u}{\partial y} = 0 \quad \text{at the origin.}$$

The proof now involves the construction of an appropriate comparison function $v=v(x, y)$.

To begin with, we observe that for arbitrary positive constants a, c, d the function

$$(4) \quad c \operatorname{arc} \sinh \left\{ \frac{\cos \frac{y}{c}}{\sinh \frac{a-x}{c}} \right\} - d$$

is a solution of the minimal surface equation in the semi-infinite rectangle Δ defined by $x < a$, $|y| < (\pi/2)c$. We take $v=v(x, y)$ to be the solution (4) with the constants adjusted so that the short side $x=a$ of Δ becomes a chord of the unit disk, as shown in Fig. 1, while also

Fig. 1

(i) the level line $v=0$ is tangent to the unit circle at the point $A=(-1, 0)$, and

(ii) $v(0)=u(0)=m$, ($m>0$).

We must of course verify that this construction is possible.

Let θ be as in Fig. 1. Then clearly

$$\cos \theta = a, \quad \sin \theta = \frac{1}{2} \pi c,$$

while (i) and (ii) require that

$$c \operatorname{arc} \sinh \left\{ \frac{1}{\sinh \frac{a+1}{c}} \right\} - d = 0 \quad \text{and} \quad c \operatorname{arc} \sinh \left\{ \frac{1}{\sinh \frac{a}{c}} \right\} - d = m.$$

Eliminating d from the last two relations and using the identity $\operatorname{arc} \sinh w = \log(w + \sqrt{1+w^2})$ yields the condition

$$m = c \log \left\{ \frac{\tanh \frac{a+1}{2c}}{\tanh \frac{a}{2c}} \right\}.$$

Expressed in terms of the parameter θ this becomes finally

$$(5) \quad m = \frac{2}{\pi} \sin \theta \cdot \log \left\{ \frac{\tanh \left(\frac{\pi}{4} \cot \frac{\theta}{2} \right)}{\tanh \left(\frac{\pi}{4} \cot \theta \right)} \right\}.$$

We shall show in the next section that the right hand side of (5) is a monotonically increasing function of θ for $0 < \theta < \pi/2$. Granting this, there then exists a unique value of θ (and consequently also unique values of a, c, d) for which the solution (4) has the properties noted above.

Let Γ be the domain between the level line $v=0$ and the chord $x=a$. We assert that Γ is entirely contained in the unit disk. To see this, it is enough to compute the curvature κ of the level line $v=0$. We find easily that

$$\kappa = \alpha \cos \frac{y}{c}$$

where y is chosen for the local coordinate on the arc, and α denotes the constant

$$\frac{\pi}{2} \csc \theta \tanh \left(\frac{\pi}{2} \cot \frac{\theta}{2} \right).$$

Since κ decreases monotonically as one follows the level line away from the point A , and since $\kappa(A) > (\pi/2) \tanh(\pi/2) > 1$, a simple geometric argument verifies the assertion.

It now follows almost exactly as in [1] that

$$(6) \quad |Vu(0)| \leq |Vv(0)|.$$

Indeed, suppose for contradiction that (6) were not true. Then in view of the normalization (3) and the fact that v is symmetric in y , we would have

$$\frac{\partial u}{\partial x} > \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}, \quad u = v \quad \text{at } 0.$$

Consider the family of functions $v(x, y)$ corresponding to values of m greater than the given one. We have observed that as m increases, so does the parameter θ . Since

$$\frac{\partial v}{\partial x} = \operatorname{csch} \frac{a}{c} = \operatorname{csch} \left(\frac{\pi}{2} \cot \theta \right) \quad \text{at } 0,$$

it follows that $\partial v / \partial x$ at 0 also increases with m . Consequently there exists a value $\bar{m} > m$ for which the corresponding function $\bar{v}(x, y)$ satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial \bar{v}}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial \bar{v}}{\partial y}, \quad u < \bar{v} \quad \text{at } 0.$$

This yields a contradiction exactly as in the earlier paper, completing the proof of inequality (6).

To obtain the conclusion of the theorem, we note that

$$(7) \quad W(0) \leq \sqrt{1 + |\nabla v(0)|^2} = \coth \left(\frac{\pi}{2} \cot \theta \right)$$

and

$$(8) \quad e^{\frac{\pi}{2} m} = \frac{\left\{ \tanh \left(\frac{\pi}{4} \cot \frac{\theta}{2} \right) \right\}^{\sin \theta}}{\left\{ \tanh \left(\frac{\pi}{4} \cot \theta \right) \right\}}.$$

In the following section it is shown that the right hand side of (7) is less than the right hand side of (8) for all values of θ in the interval $0 < \theta < \pi/2$. This completes the proof of the theorem, subject of course to the two issues which have been deferred.

2.

We show in this section that the function

$$I = \sin \theta \cdot \log \left\{ \frac{\tanh \left(\frac{\pi}{4} \cot \frac{\theta}{2} \right)}{\tanh \left(\frac{\pi}{4} \cot \theta \right)} \right\}$$

is increasing, and that the function

$$J = \coth \left(\frac{\pi}{2} \cot \theta \right) \cdot \left\{ \frac{\tanh \left(\frac{\pi}{4} \cot \theta \right)}{\tanh \left(\frac{\pi}{4} \cot \frac{\theta}{2} \right)} \right\}^{\sin \theta}$$

is everywhere less than one, where θ varies between 0 and $\pi/2$.

Let K denote the expression in braces in the formula for I . Clearly $K > 1$. Moreover, by a straightforward calculation,

$$\frac{d \log K}{d \theta} = \frac{\pi}{2} \csc^2 \theta \left\{ \frac{1}{\sinh \left(\frac{\pi}{2} \cot \theta \right)} - \frac{1 + \cos \theta}{\sinh \left(\frac{\pi}{2} \cot \frac{\theta}{2} \right)} \right\}$$

The last expression is everywhere positive, in view of the simple inequality

$$\sinh \left(\frac{\pi}{2} \cot \frac{\theta}{2} \right) > \sinh(\pi \cot \theta) > 2 \sinh \left(\frac{\pi}{2} \cot \theta \right).$$

Consequently

$$\frac{dI}{d\theta} = \cos \theta \log K + \sin \theta \frac{d \log K}{d \theta} > 0.$$

Turning to the function J , we find $J(0)=1$ and

$$\begin{aligned} \frac{dJ}{d\theta} &= J \left\{ \frac{\pi \csc^2 \theta}{\sinh(\pi \cot \theta)} - \sin \theta \frac{d \log K}{d\theta} - \cos \theta \log K \right\} \\ &< \frac{\pi J \csc^2 \theta}{2} \left\{ \frac{2}{\sinh(\pi \cot \theta)} + \frac{(1 + \cos \theta) \sin \theta}{\sinh \left(\frac{\pi}{2} \cot \frac{\theta}{2} \right)} - \frac{\sin \theta}{\sinh \left(\frac{\pi}{2} \cot \theta \right)} \right\} \\ &< \frac{\pi J \csc^2 \theta}{2 \sinh(\pi \cot \theta)} \left\{ 2 + (1 + \cos \theta) \sin \theta - 2 \sin \theta \cosh \left(\frac{\pi}{2} \cot \theta \right) \right\}. \end{aligned}$$

The quantity in braces is strictly increasing in the interval $0 < \theta \leq \pi/4$, and takes a negative value for $\theta = \pi/4$. Hence in this interval $dJ/d\theta < 0$ and $J < 1$.

We next write J in the form

$$\frac{1}{\left\{ \tanh \left(\frac{\pi}{4} \cot \frac{\theta}{2} \right) \right\}^{\sin \theta}} \cdot \frac{\left\{ \tanh \left(\frac{\pi}{4} \cot \theta \right) \right\}^{\sin \theta}}{\tanh \left(\frac{\pi}{2} \cot \theta \right)}.$$

The first factor is an increasing function of θ , while the second factor is decreasing (by an argument similar to that in the preceding paragraph). Therefore in the interval $\pi/4 \leq \theta \leq 3\pi/8$ we have

$$J \leq \frac{1}{\left\{ \tanh \left(\frac{\pi}{4} \cot \frac{3\pi}{16} \right) \right\}^{\sin \frac{3\pi}{8}}} \cdot \frac{\left\{ \tanh \frac{\pi}{4} \right\}^{V_{2/2}}}{\tanh \frac{\pi}{2}} < 0.98$$

while in the interval $3\pi/8 \leq \theta \leq \pi/2$ we have

$$J \leq \frac{1}{\tanh \frac{\pi}{4}} \cdot \frac{\left\{ \tanh \left(\frac{\pi}{4} \cot \frac{3\pi}{8} \right) \right\}^{\sin \frac{3\pi}{8}}}{\tanh \left(\frac{\pi}{2} \cot \frac{3\pi}{8} \right)} < 0.93^*.$$

This completes the proof.

3.

The estimate (2) for $W(0)$ may of course be restated as a direct inequality for $|\nabla u(0)|$, that is

$$|\nabla u(0)| \leq e^{\frac{\pi}{2}m}, \quad m = u(0).$$

A slightly better result can be obtained by noting that

$$(9) \quad |\nabla u(0)| \leq |\nabla v(0)| = \operatorname{csch} \left(\frac{\pi}{2} \cot \theta \right).$$

* A tabulation of J carried out by machine shows that the maximum of J in the interval $\pi/4 \leq \theta \leq 3\pi/8$ is approximately 0.835 and that the maximum of J in the interval $3\pi/8 \leq \theta < \pi/2$ is approximately 0.765.

A numerical calculation shows that the function

$$\operatorname{csch}\left(\frac{\pi}{2}\cot\theta\right)\left\{\frac{\tanh\left(\frac{\pi}{4}\cot\theta\right)}{\tanh\left(\frac{\pi}{4}\cot\frac{\theta}{2}\right)}\right\}^{\sin\theta}$$

is monotone increasing and everywhere less than $\frac{1}{2}\coth(\pi/4) < 0.765$. Hence from (8) and (9) we obtain

$$|\nabla u(0)| < 0.765 e^{\frac{\pi}{2}m}.$$

The coefficient here is no longer best possible, though it does seem to be fairly good.

We close with an obvious corollary of the main theorem:

Let u be a bounded solution of the minimal surface equation in a domain D of the plane. Then at any point $P \in D$

$$|\nabla u(P)| \leq e^{\frac{\pi}{2}\frac{M}{d}},$$

where $M = \max_D |u|$ and d is the distance from P to the boundary of D .

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References

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