PAUL C. FIFE

Corrigendum:
Considerations Regarding the Mathematical Basis for
Prandtl's Boundary Layer Theory

Vol. 28, pp. 184–216 (1968)

In this paper, the proofs of Lemmas 5.3 and 6.2 are inadequate as given, and there are also some technical errors in other parts of the paper. Revised proofs and corrections are given below. I wish to thank Drs. K. ŽITNÝ and M. JASNÝ of Prague for bringing these matters to my attention.

Revised Proof of Lemma 5.3. Let $W(x, \psi) = w(x, \psi) - \bar{w}(x, \psi)$. Subtracting (4.6) (with $\bar{\psi} = \psi$) from (4.5), we obtain

$$L_1 W \equiv -\sqrt{\bar{w}} W_{\psi\psi} + W_x + c(x, \psi) W = \varepsilon f(x, \eta)$$

where

$$c(x, \psi) = -\frac{w_{\psi\psi}}{w^{\frac{1}{2}} + w^{-\frac{1}{2}}} = -\frac{2R^{-1}u_{yy}}{w^{\frac{1}{2}}(w^{\frac{1}{2}} + w^{-\frac{1}{2}})}$$

and

$$\varepsilon f(x, \eta) = -2p_x(x, 0) + 2\bar{p}'(x) - \varepsilon f_3(x, \eta).$$

The proof now uses an argument based on a comparison principle (see the lemma below) for the parabolic operator L_1 .

According to Lemma 5.1 and (5.6),

$$|f(x, \eta)| \leq (2A + C_9)(1 + \eta).$$

Furthermore (5.13) yields

$$|f(x, \eta)| \leq C_{11} \text{Max}[1, \psi] \quad (5.20)$$

where $C_{11} = (2A + C_9)(2 + (3/2C_1))$. Properties (ii) and (iii) of the class $\mathcal{S}_M$ imply that

$$|R^{-1}u_{yy}| \leq R^{-1}|u_{xx} + u_{yy}| + R^{-1}|u_{xx}| \leq M + 1,$$

since $R \geq M$. Combining this with (5.10) and (5.11), we find

$$c(x, \psi) \geq -C_{12} g(\psi)^{-1} \geq -C_{12} \text{Max}[\psi^{-1}, 1] \quad (5.21)$$

where $C_{12} = 2(M + 1)/(C_3 + (C_3 C_5)^{\frac{1}{2}})$.

We define

$$\phi_1(\psi) = 4C_{10}(\psi - \alpha\psi^{\frac{1}{2}}),$$

with $\alpha > 0$ yet to be chosen. Then for any $\gamma > 0$,

$$L_1(\phi_1 e^{\gamma(x-1)}) = [\frac{3}{2}\alpha C_{10} \bar{w}^{\frac{1}{2}} \psi^{-\frac{1}{2}} + (\gamma + c)\phi_1] e^{\gamma(x-1)}.$$

From (5.11) and (5.21) it follows that the last expression is

$$\geq 4C_{10}[\frac{3}{2}C_5^{\frac{1}{2}}\alpha g^{\frac{1}{2}}(\psi)\psi^{-\frac{1}{2}} - C_{12}]e^{\gamma(x-1)} \geq 4C_{10}(\frac{3}{2}C_5^{\frac{1}{2}}\alpha - C_{12})e^{\gamma(x-1)}$$

for $\psi \leq \frac{1}{\alpha^2}$, $x \geq 1$. We choose $\alpha = 2(C_{11} + 4C_{10}C_{12})/3C_{10}C_5^{\frac{1}{2}} > 1$, whence

$$L_1(\phi_1 e^{\gamma(x-1)}) \geq C_{11}$$

for $\psi \leq 1$, $x \geq 1$. Now define

$$\phi_2(\psi) = C_{10}(\psi + (2\alpha)^{-2})$$

so that $\phi_2 = \phi_1$ and $\phi_2' = \phi_1'$ at $\psi = (2\alpha)^{-2}$. Also

$$\begin{aligned} L_1(\phi_2 e^{\gamma(x-1)}) &= 4C_{10}(\gamma + c)\phi_2 e^{\gamma(x-1)} \\ &\geq C_{10}\psi(\gamma - C_{12} \text{Max}[\psi^{-1}, 1])e^{\gamma(x-1)} \geq C_{11} \text{Max}[1, \psi] \end{aligned}$$

for $\psi \geq (2\alpha)^{-2}$, provided that $\gamma = (4\alpha^2 C_{11}/C_{10}) + C_{12}$.

We now define

$$\Phi(x, \psi) = 4C_{10}e^{\gamma(x-1)} \begin{cases} \psi - \alpha\psi^{\frac{1}{2}}, & \psi \leq (2\alpha)^{-2} \\ \frac{\psi}{4} + \frac{(2\alpha)^{-2}}{4}, & \psi \geq (2\alpha)^{-2}. \end{cases}$$

The above inequalities then yield

$$L_1 \Phi \geq C_{11} \text{Max}[1, \psi] \geq |f| = \varepsilon^{-1} |L_1 W|.$$

We have thus shown that

$$L_1(\varepsilon\Phi \pm W) \geq 0$$

for all $x > 1$, $\psi > 0$.

The comparison statement we need is the following:

Lemma. For some $\psi_1 > 0$ let $\mathcal{D}_1$ be the domain $\{(x, \psi) : 1 < x < 2, 0 < \psi < \psi_1\}$, and let Γ be the part of the boundary of $\mathcal{D}_1$ on which $x = 1$, $\psi = 0$, or $\psi = \psi_1$. Let $V(x, \psi)$ be a function continuous in $\overline{\mathcal{D}_1}$ with derivative V_ψ continuous in $\mathcal{D}_1$; suppose also that $V_{\psi\psi}$ and V_x are continuous in $\mathcal{D}_1$ with respect to x , and that $V_{\psi\psi}$ and V_x are piecewise continuous in $\mathcal{D}_1$ with respect to ψ . Furthermore assume that $L_1 V \geq 0$ in $\mathcal{D}_1$. Under these circumstances, if $V \geq 0$ on Γ , then $V \geq 0$ in $\mathcal{D}_1$.

Proof. For positive constants β , a_0 , and c_0 (to be chosen later) and for all $\delta > 0$, we define

$$V_\delta(x, \psi) \equiv V(x, \psi) + \delta e^{\beta x} \left(\frac{8\psi^{\frac{1}{2}}}{a_0} + \frac{1}{c_0} \right).$$

Then

$$\begin{aligned} LV_\delta &= LV - \bar{w}^\pm \left\{ \delta e^{\beta x} \frac{8}{a_0} \left(-\frac{1}{4} \right) \psi^{-\frac{1}{2}} \right\} + (\beta + c) \delta e^{\beta x} \left(\frac{8\psi^{\frac{1}{2}}}{a_0} + \frac{1}{c_0} \right) \\ &= LV + \delta e^{\beta x} \left\{ (\beta + c) \left(\frac{8}{a_0} \psi^{\frac{1}{2}} + \frac{1}{c_0} \right) + \frac{2\bar{w}^\pm}{a_0} \psi^{-\frac{1}{2}} \right\} \\ &\geq \delta e^{\beta x} \left\{ \frac{\beta}{c_0} - |c| \left(\frac{8}{a_0} \psi^{\frac{1}{2}} + \frac{1}{c_0} \right) + 2 \left(\frac{\bar{w}^\pm}{a_0 \psi^{\frac{1}{2}}} \right) \psi^{-1} \right\}. \end{aligned}$$

Now choose a_0 so that $\bar{w}^\pm/a_0 \psi^{\frac{1}{2}} \geq 1$ in $\mathcal{D}_1$ and c_0 so that $c \geq -c_0 \psi^{-1}$ in $\mathcal{D}_1$. This is clearly possible because of (5.11) and (5.21). Then

$$\begin{aligned} LV_\delta &\geq \delta e^{\beta x} \left\{ \frac{\beta}{c_0} - \frac{c_0}{4} \left(\frac{8}{a_0} \psi^{\frac{1}{2}} + \frac{1}{c_0} \right) + 2 \psi^{-1} \right\} \\ &= \delta e^{\beta x} \left\{ \frac{\beta}{c_0} + \left[1 - \frac{8c_0}{a_0} \psi^{\frac{1}{2}} \right] \psi^{-1} \right\}. \end{aligned}$$

The quantity in braces is positive for all $\beta \geq 0$ and for sufficiently small ψ , and is independent of δ . We now choose β so large that this quantity is in fact positive for all $\psi > 0$. These $LV_\delta > 0$ in $\mathcal{D}_1$ for all $\delta > 0$.

For any $\delta > 0$, we have $V_\delta > 0$ on Γ . By the (uniform) continuity of V in $\overline{\mathcal{D}_1}$, there is a small positive number ψ_0 (depending on δ) such that $V_\delta > 0$ for $0 \leq \psi \leq \psi_0$, $1 \leq x \leq 2$. In the region $\overline{\mathcal{D}_2} = \{(x, \psi) : \psi_0 \leq \psi \leq \psi_1, 1 \leq x \leq 2\}$, L is uniformly positive and c is bounded from below. Hence by a standard comparison theorem $V_\delta > 0$ in $\overline{\mathcal{D}_2}$ and hence in $\overline{\mathcal{D}_1}$. Letting $\delta \rightarrow 0$, we obtain the desired result.

We now apply the above lemma to the function $V = \varepsilon \Phi \pm W$, with $\psi_1 = (2+2b)/C_{10}\varepsilon$. (It may be checked from (5.5'), (5.13) that $y \leq 1$ in this strip, so we remain in $\mathcal{D}'$.) It must be verified that $\varepsilon \Phi \pm W \geq 0$ on Γ . This is true for $\psi = 0$, since $\Phi = W = 0$. Also for $\psi = (2+2b)/C_{10}\varepsilon$,

$$\varepsilon \Phi \geq \varepsilon(4C_{10})(2+2b)/4C_{10}\varepsilon = 2+2b \geq |w| + |\bar{w}|,$$

the last inequality following from (i) and (5.12). But the right side clearly exceeds $|W|$, so $\varepsilon \Phi \pm W \geq 0$ here also. Finally for $x = 1$, we use Lemma 5.2: $\varepsilon \Phi \geq \varepsilon C_{10} \psi \geq |W(1, \psi)|$.

Thus the lemma applies, and we obtain

$$|W(x, \psi)| \leq \varepsilon \Phi \leq 4\varepsilon C_{10} e^{\gamma} \psi = \varepsilon C_{14} \psi$$

where $C_{14} = 4C_{10}e^{\gamma}$. This proves Lemma 5.3.

Revised Proof of Lemma 6.2. We differentiate (6.7) twice with respect to ψ and, for fixed δ , set $Z = -w_\psi^\delta$. (The theory of OLEINIK & KRUKOV [7] shows that w^δ has sufficient regularity in the interior to carry out the differentiation.) We then easily obtain the following equation for Z :

$$(w^\delta)^{\frac{1}{2}} Z_{\psi\psi} - Z_x + (w^\delta)^{-\frac{1}{2}} w_\psi^\delta Z_\psi - \frac{1}{4} (w^\delta)^{-\frac{1}{2}} (w_\psi^\delta)^2 Z - \frac{1}{2} (w^\delta)^{-\frac{1}{2}} Z^2 = 0.$$

We shall apply a comparison argument, but must first investigate the properties of Z on or near the boundary. This boundary (excluding $\{x=2\}$) consists of the three segments $x=\delta$, $\psi=\delta^{-1}$, and $\psi=0$. As $x\downarrow\delta$ with $\psi>0$, we have $Z\rightarrow -w_{\psi}^1\psi\geq 0$; this follows because of regularity up to the boundary of solutions of regular quasilinear parabolic problems; see [7]. As $\psi\uparrow\delta^{-1}$, we obtain

$$\begin{aligned} Z &= -(w_x^\delta + 2p'_\delta)(w^\delta)^{-\frac{1}{2}} \rightarrow -(w'_x + 2p'_\delta)(w^1)^{-\frac{1}{2}} \\ &= -(w_x^1 + 2(-a_1x))(w^1)^{-\frac{1}{2}} + 2(-a_1x - p'_\delta)(w^1)^{-\frac{1}{2}} \\ &= -w_{\psi\psi}^1(x, \delta^{-1}) + 2(-a_1x - p'_\delta)(w^1)^{-\frac{1}{2}}. \end{aligned}$$

That the first term is ≥ 0 is a known property of w^1 ; the second is ≥ 0 due to our construction of p'_δ . Thus $Z\geq 0$ for $\psi=\delta^{-1}$.

Finally, as $\psi\rightarrow 0$ we know from Theorem 4.1 that $w_x^\delta\rightarrow 0$. Hence $(w^\delta)^{\frac{1}{2}}w_{\psi\psi}^\delta - 2p'_\delta(x)\rightarrow 0$. This implies that $(w^\delta)^{\frac{1}{2}}Z\rightarrow -2p'_\delta(x)>0$. Therefore for some small enough number $\beta>0$ we have $Z(x, \beta)>0$.

In all, we have $Z\geq 0$ for $\psi=\beta$, $\psi=\delta^{-1}$, and $x=\delta$. These three lines, together with $\{x=2\}$, bound a region in which Z satisfies an equation of the form

$$w^{\frac{1}{2}}Z_{\psi\psi} - Z_x + B(x, \psi)Z_\psi + C(x, \psi)Z = 0$$

with bounded coefficients B and C . Thus we conclude that $Z\geq 0$ in this region. Since this is true for all δ and all small $\beta>0$, it is true of the limit function $-\bar{w}_{\psi\psi}$ in the entire domain $\psi>0$, $x>0$. But $\bar{u}_{\eta\eta} = \sqrt{\bar{w}}\bar{w}_{\psi\psi} < 0$. The lemma is thereby proved.

Corrections

Page 187, (2.5): $u_x + v_y = 0$.

Page 188, 1.9⁻: ... $(a/8)^{\frac{1}{2}} \geq M$, ...

In the proofs of Theorems 2.2 and 3.2, the definition of $\mathcal{D}'''$ should be revised to include only that part of the given rectangle for which $u\geq 0$.

Page 190, 1.12, 13; page 210, 1.13: Replace $a_1/16$ by $a_1/6$ in the definition of M^* .

Page 190, 1.15; page 191, 1.3; page 210, 1.13: Replace $15M^*$ by $20M^*$.

Page 190, 1.16: Adjoin the assumption $RM^*\geq 1$.

Page 190, 1.9⁻: $|\Phi_x|\leq (16/3)U_1$, $|\Phi_{xx}|\leq (8/3)xU_1$. (Corresponding changes in the last 3 lines.)

Page 191, Theorem 3.2: Add the assumptions $u(0, y)=v(x, 0)=0$, $b_1>0$, and $R\geq 3$.

Page 191, (3.6): $\Phi_{xx}\leq 6Kx$. (Corresponding changes, lines 10⁻, 9⁻, 8⁻.)

Page 191, 1.2⁻, 1⁻: insert x_1 after K and k^* .

Page 192, 1.5: ... $\phi^2(\infty)$.

Page 193, (4.5): insert a factor 2 before first ε .

Page 193, (4.6): Replace $+2\bar{p}_x$ by $-2\bar{p}_x$.

Page 194, 1.7⁻: ... Let $-b\leq p'(x)\leq -a<0$,

Page 195, 1.8⁻: $\bar{w}(0, \psi)\leq \dots$

Page 197, l.13: (ii) u and v and their x -derivatives up to order 2 ...

Page 197, (5.1): $|f_3(x, \eta)| \leq \dots$

It is stated at the top of page 198 that $|\phi'(\eta)| \leq M$. Rather than this, it follows from the definition of $\mathcal{S}_M$ that $|\phi''(\eta)| \leq M(1 + \varepsilon^2) \leq 2M$, and from a standard interpolation inequality [5, p. 114] that $|\phi'(\eta)| \leq 4\sqrt{M}$. Because of this, the constant in Lemma 5.2 should be changed to $C_{10} = 3A(2C_2 + 4\sqrt{M})/C_1$, and (5.17) to $|\bar{w}_{\bar{\psi}}| \leq 8\sqrt{M}$.

Page 198, l.4: $C_1 = a/20M^*$, ...

Page 199, (5.18): $\dots \leq \varepsilon A \int_0^\eta \text{Min}[\eta', 1] d\eta'$. (Similar change, line 4⁻.)

Page 203, l.6: $\dots \text{Max}[\bar{\psi}^\frac{1}{2}, \bar{\psi}^2]$.

Page 203, l.16: Replace C_5 by $C_5^\frac{1}{2}$.

Page 205, (6.3): Replace $4/5$ by $3/4$. This results in changes on page 208: l.8⁻: $C_5 = (9a_1^\frac{1}{2}/16) \text{Min}[1, 8/5a_1^\frac{1}{2}]$, and $C_6 = (25b_1^\frac{1}{2}/16) \text{Max}[1, 8/3b_1^\frac{1}{2}]$.

Page 208, l.5⁻: $\frac{3}{4} \text{Min}[(8t/5)^\frac{1}{2}, 1] \leq f'(f^{-1}(t)) \leq \frac{5}{4} \text{Min}[(8t/3)^\frac{1}{2}, 1]$.

Page 205, l.7⁻: $-\sqrt{\bar{w}^0} w_{\bar{\psi}\psi}^0 \dots$

Page 206, (6.7): $\dots - 2p'_\delta \dots$

Page 206, l.13: Replace w^2 by w^1 ; add the condition $w^\delta(x, 0) = 0$.

Page 206, l.15: $\bar{p}'(x) \leq p'_\delta(x) \leq -a_1 x$, $p'_\delta(\delta) = -a_1 \delta$, ...

Page 206, l.14⁻: $\dots N\phi(x, \psi) \equiv -\sqrt{\bar{\phi}} \dots$

Page 206, l.13⁻: $\dots Nw^1 = 2a_1 x \leq \dots$

Page 207, l.14: $0 < x < 2$, $\eta > 0$.

Page 211, l.1: $\dots \bar{c}(x, \psi) = -2R^{-1} \bar{u}_{yy}/(\bar{w} + (w\bar{w})^\frac{1}{2})$, and $\varepsilon f = -2p_x(x, 0) + 2\bar{p}^1(x) - \varepsilon f_3$.

Page 211, l.5, 8: Replace A by $2A$.

Department of Mathematics
University of Arizona
Tucson, Arizona 85721

(Received November 24, 1971)