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Corrigendum:
A Nonlinear Integral Equation
with Applications to Neutron Transport Theory

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The proof of Theorem 5 is incorrect. Also another condition must be added to the hypothesis:

Theorem 5. *Let $c > c^*$ and E and G satisfy $E_1 - E_5$ and $G_1 - G_4$. If, furthermore, E is symmetric and satisfies E_6 , then (5) possesses a unique nontrivial solution in $\mathcal{S}$.*

Proof. By Theorems 1 and 3 it suffices to prove $\underline{\varphi} \equiv \bar{\varphi}$. Both $\underline{\varphi}$ and $\bar{\varphi}$ satisfy (5). Therefore, using the symmetry of E ,

$$\begin{aligned} \int_{\mathcal{D}} [\bar{\varphi}(x) G(x, \underline{\varphi}(x)) - \underline{\varphi}(x) G(x, \bar{\varphi}(x))] dx &= \int_{\mathcal{D}} \int_{\mathcal{D}} E(x, y) [G(y, \bar{\varphi}(y)) G(x, \underline{\varphi}(x)) \\ &\quad - G(y, \underline{\varphi}(y)) G(x, \bar{\varphi}(x))] dx dy \\ &= 0 = \int_{\mathcal{D}} \bar{\varphi}(x) \underline{\varphi}(x) \left[\frac{G(x, \underline{\varphi}(x))}{\underline{\varphi}(x)} - \frac{G(x, \bar{\varphi}(x))}{\bar{\varphi}(x)} \right] dx. \end{aligned}$$

Since $\bar{\varphi} \underline{\varphi} > 0$ in $\mathring{\mathcal{D}}$, and since by (27) the term in brackets is positive somewhere in $\mathring{\mathcal{D}}$ unless $\underline{\varphi} \equiv \bar{\varphi}$, the vanishing of the integral implies $\underline{\varphi} \equiv \bar{\varphi}$.

We are unable to prove Theorem 5 if the symmetry condition on E is removed. However, we have:

Theorem 5'. *Let $c > c^*$ and E and G satisfy $E_1 - E_5$ and $G_1 - G_4$. If, furthermore, $E > 0$ in $\mathcal{D} \times \mathcal{D}$, then (5) possesses a unique nontrivial solution in $\mathcal{S}$.*

Proof. If $\underline{\varphi} \not\equiv \bar{\varphi}$, then $E > 0$ in $\mathcal{D} \times \mathcal{D}$ implies $0 < \delta \leq \underline{\varphi} < \bar{\varphi}$ in $\mathcal{D}$ where δ is a constant. Let $t \in (0, 1)$ and $\Phi = t \underline{\varphi} + (1-t) \bar{\varphi}$. Then, since G is strictly concave:

$$\begin{aligned} N(\Phi) &= N(t \underline{\varphi} + (1-t) \bar{\varphi}) = \int_{\mathcal{D}} E(x, y) G(y, t \underline{\varphi}(y) + (1-t) \bar{\varphi}(y)) dy \\ &> \int_{\mathcal{D}} E(x, y) [t G(y, \underline{\varphi}(y)) + (1-t) G(y, \bar{\varphi}(y))] dy = t N(\underline{\varphi}) + (1-t) N(\bar{\varphi}) = \Phi. \end{aligned}$$

By the same argument, if $s > 1$ and $s \underline{\varphi} \in \mathcal{S}$ then

$$N(s^{-1} s \underline{\varphi} + (1-s^{-1}) 0) = N(\underline{\varphi}) = \underline{\varphi} > s^{-1} N(s \underline{\varphi}),$$

i.e., $N(s\underline{\varphi}) < s\underline{\varphi}$. Choose $s > 1$ so that $s\underline{\varphi} \leq \overline{\varphi}$, and choose $t > 0$ so that $\Phi = t\underline{\varphi} + (1-t)\overline{\varphi} \leq s\underline{\varphi}$, with equality at at least one point x_0 . Then

$$s\underline{\varphi}(x_0) > N(s\underline{\varphi})(x_0) \geq N(\Phi)(x_0) > \Phi(x_0),$$

a contradiction. Thus $\overline{\varphi} \equiv \underline{\varphi}$.

Note that no uniform lower bound away from 0 is assumed for E in Theorem 5'.

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