Addendum to “Some examples of nonuniqueness of solutions to the equation of linear elastodynamics”

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1.

The present paper is meant as nothing more than a short note, devoted to complete the picture of nonuniqueness in linear elastodynamics given in the work quoted in the title [1]. It was originally planned as a “note added in proof” to [1], but when I received the galley proofs of [1], it was not yet ready. According to its nature of an “appendix” to [1], throughout this note we shall always refer to the notations and the formulae of the full paper. The reader should be then aware that, when quoting formula (*) or Section X or Remark Y, we mean formula (*) or Section X or Remark Y of [1].

The need of this addition to [1] may be easily understood by considering the following objection to the counter-examples given there.

The one-dimensional model of the dynamical problem of linear elasticity is not quite suitable to sample the considered case in which a point $\mathbf{o}$ of the body B is allowed to be a singularity point for the material data. Indeed, the statement of the boundary-initial value problem, in its general form, requires that B is an “open connected set” (cf. Section 3); now, when dealing with a one-dimensional example, the presence of the singularity point $\mathbf{o}$ “breaks” the connection of B , which in this case turns out to be the join of two different bodies $B_1 = (-x_0, 0)$ and $B_2 = (0, +\infty)$ (cf. Section 5). Thus, at a first glance, it could seem that if we would construct an analogous example in the general case (i.e. in two or three dimensions), we should think, rather than of a *single* body B containing a singularity point $\mathbf{o}$, of two different bodies B_1 and B_2 with a common boundary γ , where the material data are allowed to be singular. In such a problem, the dependence of the uniqueness of solutions on some conditions on γ (e.g., the continuity of the displacement and/or the stress field across γ , or the validity of the Third Law of Dynamics for the contact interactions between B_1 and B_2) is to be

expected even when the data are assumed to have only jump discontinuities across γ , so that an example of nonuniqueness says nothing essentially new.

2.

In view of the above remark, it will be important to extend the examples given in Sections 5 and 6, at least, to the case of a two-dimensional body.

To this aim, let B be a two-dimensional *isotropic* elastic body, identified (in its reference configuration) with the (x^1, x^2) -plane of an assigned reference frame $\mathcal{R}$. Assume

- 1) a system of polar coordinates (r, θ) is introduced on B ;
- 2) the singularity point for the material data of B is located at the pole $\mathbf{o}$ of (r, θ) ;
- 3) the *Lamé moduli* λ and μ of B satisfy the conditions

$$\lambda = 0, \quad \mu = \mu(r); \quad (41)$$

- 4) the *density field* ϱ of B obeys the relation

$$\varrho = \varrho(r). \quad (42)$$

In order to show that the motion of B is *not* uniquely determined by the body forces acting on B and the boundary and initial condition, it is sufficient to find nonzero solutions to System (8)–(9)₃, with

$$\mathbf{b} = \mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}. \quad (43)$$

(recall that $\partial B = \emptyset$). Let us then write System (8) in the cylindrical coordinates (r, θ, z) , and look for solutions $\mathbf{u} = (u_r, u_\theta, u_z)$ such that

$$u_r = u_r(r, t), \quad u_\theta = u_\theta(r, t), \quad u_z = u_z(r, t). \quad (44)$$

Because of condition (41)₁, System (8) takes the form

$$\left. \begin{aligned} \varrho \ddot{u}_r &= \partial[2\mu(\partial u_r / \partial r)] / \partial r + (2\mu/r)[(\partial u_r / \partial r) - (u_r/r)], \\ \varrho \ddot{u}_\theta &= \partial[\mu((\partial u_\theta / \partial r) - (u_\theta/r))] / \partial r + (2\mu/r)[(\partial u_\theta / \partial r) - (u_\theta/r)] \\ \varrho \ddot{u}_z &= \partial[\mu(\partial u_z / \partial r)] / \partial r + (\mu/r)(\partial u_z / \partial r) \end{aligned} \right\} \text{ on } Q, \quad (45)$$

where $Q = \{(r, t) \in \mathbb{R}^2 : r > 0, t \geq 0\}$.

For the sake of simplicity, we shall look for nonzero solutions $\mathbf{u}$ such that $u_r(r, t) = 0$, so that we can confine ourselves to solve $(45)_2$ and $(45)_3$. A straightforward calculation shows that they are equivalent to the equations

$$\left. \begin{aligned} (r^3 \varrho)(\ddot{u}_0/r) &= \partial[(r^3 \mu)(\partial(u_0/r)/\partial r)]/\partial r, \\ (r\varrho)\ddot{u}_z &= \partial[(r\mu)(\partial u_z/\partial r)]/\partial r \end{aligned} \right\} \text{ on } Q. \quad (46)$$

Assume now

$$r\varrho = q'(r), \quad r\mu = [q'(r)]^{-1}, \quad (47)$$

where q is the function introduced in assumption b) of Section 3. Then equation $(46)_2$ becomes

$$q'(r)\ddot{w} = \partial\{[q'(r)]^{-1}(\partial w/\partial r)\}/\partial r \quad (48)$$

(with $w = u_z$) and can be solved in the same way as equation (31) (cf. [2]), thus leading to the solution

$$u_z(r, t) = \begin{cases} 0 & \text{on } Q^- = \{(r, t) \in Q : t \leq q(r)\}, \\ u_s(t - q(r)) & \text{on } Q^+ = \{(r, t) \in Q : t > q(r)\}, \end{cases} \quad (49)$$

where

$$u_s(t) = u_z(0, t) \quad t \geq 0, \quad (50)$$

and $u_s(0) = 0$. In conclusion, under assumptions (47), System (8)–(9)₃–(43) admits infinitely many nonzero solutions $\mathbf{u} = (0, 0, u_z(r, t))$ depending on the “additional datum” (50).

It is readily seen (for details, cf. [3]) that, at least when $q'(r) = O(r)$ and $q''(r) = O(1)$, example (49) is sufficiently regular to be a “genuine” counterexample to uniqueness (in the sense of Remark 1). Unfortunately, it could be considered unsatisfactory because the displacement is at each instant *perpendicular* to the plane of (the reference configuration of) the body: in a genuine two-dimensional scheme, such displacements should not be considered. So we look for more convincing examples. If we assume

$$r^3 \varrho = q'(r), \quad r^3 \mu = [q'(r)]^{-1} \quad (51)$$

and put

$$u_0 = rU_0,$$

we may write equation (46)₁ in the same form (48) with $w = U_0$, and immediately arrive at the solution

$$u_0(r, t) = \begin{cases} 0 & \text{on } Q^-, \\ rU_s(t - q(r)) & \text{on } Q^+, \end{cases} \quad (52)$$

where again

$$U_s(t) = U_0(0, t) \quad t \geq 0. \quad (53)$$

We may then conclude again that, under assumptions (51), System (8)–(9)₃–(43) admits infinitely many nonzero solutions $\mathbf{u} = (0, u_0(r, t), 0)$ depending on the “additional datum” (53). It should be stressed that, in this case,

$$u_0(0, t) = 0 \quad t \geq 0 \quad (54)$$

whatever the “datum” (53) may be.

Bearing in mind Remark 1, we must however recall that, no matter whether the material data are regular at $\mathbf{o}$ or not, then the nonzero solutions to System (46)₁–(54) *cannot be of class* $C^1(B \times (0, +\infty))$. In view of Remark 1, we cannot then take them as a genuine example of the link between nonuniqueness and the behaviour of the material data at $\mathbf{o}$. Moreover, from a purely mathematical viewpoint, the following objection could be expected: when Problem (8)–(9)₃–(43) is written in polar coordinates in the whole plane and is solved *in the class of solutions depending only on the distance from $\mathbf{o}$* , and either conditions (47) or conditions (51) are assumed to hold, then we are always led to formulation (48) with the initial conditions

$$w(r, 0) = 0, \quad \dot{w}(r, 0) = 0 \quad r \geq 0. \quad (55)$$

These conditions cannot be expected to be sufficient to determine uniquely the solution: in order to be sure that System (48)–(55) admits only the null solution, we need either

- i) to assign data (55) also for $r < 0$ (but this is absurd, in view of the meaning of r) or
- ii) to assign the value of the unknown function $w(r, t)$ on the t -axis.

According to this objection, we could suppose that nonuniqueness is a consequence of looking for axially symmetric solutions to System (8)–(9)₃–(43), rather than of the behaviour of the material data q and μ at $\mathbf{o}$.

We then find ourselves again in the need to look for better examples.

3.

Let us now go back to System (8) in cartesian coordinates. We assume again relations (41)₂–(42) to be met, but we suppose – as we did in the example given in Remark 1 –

$$\lambda = \lambda(r) = -\mu(r). \quad (56)$$

We then look for solutions $\mathbf{u} \equiv (u, 0, 0)$, such that $u(x^1, x^2) = u(r(x^1, x^2))$.

Under our assumptions, System (8) is reduced to the equation

$$q\ddot{u} = \partial[\mu(\partial u/\partial x)]/\partial x + \partial[\mu\partial u/\partial y]/\partial y \quad \text{on } \{(x, y, t): x \neq 0, y \neq 0, t > 0\},$$

where we have set $x^1 = x$ and $x^2 = y$. A straightforward calculation now shows that this last equation is equivalent to

$$rq\ddot{u} = \partial[r\mu(\partial u/\partial r)]/\partial r.$$

Hence, if we assume again relations (47) to be met, we have

$$\ddot{u} = [q'(r)]^{-1} \partial\{[q'(r)]^{-1}(\partial u/\partial r)\}/\partial r.$$

As is well known [2], the general solution of this last equation is of the form

$$\begin{aligned} u(r(x, y), t) &= (1/2) \left\{ u_0(q^{-1}(q(r) - t)) \right. \\ &\quad \left. + u_0(q^{-1}(q(r) + t)) + \int_{q^{-1}(q(r)-t)}^{q^{-1}(q(r)+t)} u_0(s)q'(s) ds \right\}, \quad (t \leq q(r)) \\ u(r(x, y), t) &= u_s(t - q(r)) + (1/2) \left\{ u_0(q^{-1}(q(r) + t)) - u_0(q^{-1}(t - q(r))) \right. \\ &\quad \left. + \int_{q^{-1}(t-q(r))}^{q^{-1}(q(r)+t)} u_0(s)q'(s) ds \right\}, \quad (t \geq q(r)) \end{aligned} \quad (57)$$

where $u_0(r)$ and $u_0(r)$ are the *initial data* and $u_s(t)$ is again an *additional datum*, introduced by the condition

$$u(0, t) = u_s(t), \quad t \geq 0.$$

As a consequence, if the initial data are identically zero, the solution is given by

$$u(r(x, y), t) = \begin{cases} 0 & \text{for any } r \geq 0 \text{ and for any } t \leq q(r) \\ u_s(t - q(r)) & \text{for any } r \geq 0 \text{ and for any } t \geq q(r). \end{cases} \quad (58)$$

It is now easy to verify that this solution is actually twice continuously differentiable in the whole $\mathbb{R}^2 \times [0, +\infty)$, at least for some proper choice of the function $q(r)$. Indeed, since

$$\begin{aligned} (\partial u / \partial x) &= (x/r)q'(r)u'_s(t - q(r)), \\ (\partial u / \partial y) &= (y/r)q'(r)u'_s(t - q(r)), \\ (\partial^2 u / \partial x^2) &= (1/r)q'(r)u'_s(t - q(r)) - (x^2/r^3)q'(r)u'_s(t - q(r)) \\ &\quad + (x^2/r^2)q''(r)u'_s(t - q(r)) + [(x/r)q'(r)]^2 u''_s(t - q(r)), \\ (\partial^2 u / \partial y^2) &= (1/r)q'(r)u'_s(t - q(r)) - (y^2/r^3)q'(r)u'_s(t - q(r)) \\ &\quad + (y^2/r^2)q''(r)u'_s(t - q(r)) + [(y/r)q'(r)]^2 u''_s(t - q(r)), \\ (\partial^2 u / \partial x \partial y) &= (\partial^2 u / \partial y \partial x) = -(xy/r^3)q'(r)u'_s(t - q(r)) \\ &\quad + (xy/r^2)q''(r)u'_s(t - q(r)) + (xy/r^2)[q'(r)]^2 u''_s(t - q(r)), \end{aligned}$$

it is readily seen that, provided $q'(r) = O(r)$ and $q''(r) = O(1)$, the solution $\mathbf{u} = (u, 0, 0)$, with u given by (58), turns out to be of class C^2 in the whole $\mathbb{R}^2 \times (0, \infty)$.

References

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3. B. Carbonaro and R. Russo, Singularity problems in linear elastodynamics, to appear.