What Kind of Insight Provide Analytical Solutions of Quantum Models?

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Abstract There are several concepts of what constitutes the analytical solution of a quantum model, as opposed to the mere “numerically exact” one. This applies even if one considers only the determination of the discrete spectrum of the corresponding Hamiltonian, setting aside such important questions as the asymptotic dynamics for long times. In the simplest case, the spectrum can be given in closed form, the eigenvalues E_j , $j = 0, \dots, N \leq \infty$ read $E_j = f(j, \{p_k\})$, where f is a known function of the label $j \in \mathbb{N}_0$ and the $\{p_k\}$ are a set of numbers parameterizing the Hamilton operator. This kind of solution exists only in cases where the classical limit of the model is Liouville-integrable. Some quantum-mechanical many-body systems allow the determination of the spectrum in terms of auxiliary parameters $[\{k_j\}, \{n_l\}]$ as $E(\{n_l\}) = f(\{k_j(\{n_l\})\})$ where the $\{k_j(\{n_l\})\}$ satisfy a coupled set of transcendental equations, following from a certain ansatz for the eigenfunctions. These systems (integrable in the sense of Yang-Baxter (Eckle 2019)) may have a Hilbert space dimension growing exponentially with the system size L , i.e., $N \sim e^L$. The simple enumeration of the energies with the label j is replaced by the multi-index $\{n_l\}$. Although no priori knowledge about the spectrum is available, its statistical properties can be computed exactly (Berry and Tabor 1977). Other integrable and also non-integrable models exist where N depends polynomially on L and the energies E_j are the zeroes of an analytically computable transcendental function, the so-called G -function $G(E, \{p_k\})$ (Braak 2013a, 2016), which is proportional to the spectral determinant. Although no closed formula for E_j as function of the index j exists, detailed qualitative insight into the distribution of the eigenvalues can be obtained (Braak 2013b). Possible applications of these concepts to information compression and cryptography are outlined.

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