Number Theoretic Study in Quantum Interactions

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Abstract The quantum interaction models, with the quantum Rabi model as a distinguished representative, are recently appearing ubiquitously in various quantum systems including cavity and circuit quantum electrodynamics, quantum dots and artificial atoms, with potential applications in quantum information technologies including quantum cryptography and quantum computing (Haroche and Raimond 2008; Yoshihara et al. 2018). In this extended abstract, based on the contents of the talk at the conference, we describe shortly certain number theoretical aspects arising from the *non-commutative harmonic oscillators* (NCHO: see Parmeggiani and Wakayama 2001; Parmeggiani 2010) and *quantum Rabi model* (QRM: see Braak 2011 for the integrability) through their respective spectral zeta functions.

The quantum interaction models, with the quantum Rabi model as a distinguished representative, are recently appearing ubiquitously in various quantum systems including cavity and circuit quantum electrodynamics, quantum dots and artificial atoms, with potential applications in quantum information technologies including quantum cryptography and quantum computing (Haroche and Raimond 2008; Yoshihara et al. 2018). In this extended abstract, based on the contents of the talk at the conference, we describe shortly certain number theoretical aspects arising from the *non-commutative harmonic oscillators* (NCHO: see Parmeggiani and Wakayama 2001; Parmeggiani 2010) and *quantum Rabi model* (QRM: see Braak 2011 for the integrability) through their respective spectral zeta functions.

In physics, given a quantum interaction model, one of the main interests is to know the heat kernel (or equivalently the evolution operator) since, among other reasons, the heat kernel gives the partition function by taking the trace. With partition function of the model, we may also get the analytic properties of the spectral zeta function

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by means of the Mellin transform. A spectral zeta function is defined, in general, as the Dirichlet series formed by the spectrum (eigenvalues) of the corresponding Hamiltonian (Ichinose and Wakayama 2005; Sugiyama 2018). Notice that knowing the spectral zeta function is essentially equivalent to knowing the partition function in any quantum system.

In the case of the NCHO, the Hamiltonian is given by

$$\mathcal{Q} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(x \frac{d}{dx} + \frac{1}{2} \right),$$

with $\alpha, \beta > 0$ and $\alpha\beta > 1$ (the condition for having only a discrete spectrum with positive eigenvalues), and the spectral zeta function by

$$\zeta_{\mathcal{Q}}(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} \quad (\Re(s) > 1),$$

where $(0 <) \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots (\nearrow \infty)$ are the eigenvalues of NCHO. Note that the lowest eigenstate is multiplicity free (Hiroshima and Sasaki 2014) and the multiplicity of general eigenstate is less than or equal to two (Wakayama 2016). The function $\zeta_{\mathcal{Q}}(s)$ is meromorphically continued to the whole complex plane with a unique simple pole at $s = 1$ and has trivial zeros at the even non-positive integers (Ichinose and Wakayama 2005). Although our study is very much influenced by the classical algebro-geometric work on Apéry numbers for the Riemann zeta function in Beukers (1987) and its subsequent developments, since the family of generating functions for *Apéry-like numbers* (Kimoto and Wakayama 2006) arising via the NCHO possesses a remarkable hierarchical structure, there is a decisive difference between these two (Ichinose and Wakayama 2005; Kimoto and Wakayama 2019).

For instance, there are congruence properties of the (normalized) Apéry-like numbers that have arisen naturally from the special values $\zeta_{\mathcal{Q}}(2)$ at $s = 2$. This can be seen by the same idea that guided the studies for the Apéry numbers for $\zeta(2) (= \pi^2/6)$ in Beukers (1985). These congruence properties led us further to observe that the generating function w_2 of the Apéry-like numbers for $\zeta_{\mathcal{Q}}(2)$ is interpreted as a $\Gamma(2)$ -modular form of weight 1 (Kimoto and Wakayama 2007) in the same way as in a pioneering study by Beukers (1983, 1987) for the Apéry numbers. It is worth mentioning that the recurrence equation of these Apéry-like numbers defined in Kimoto and Wakayama (2006) provides one of the particular examples listed in Zagier (2009) (it gives #19 in the list).¹ Also, recently, certain congruence relations among these Apéry-like numbers conjectured in Kimoto and Wakayama (2006) resembling Rodriguez-Villegas type congruences (Mortenson 2003) were proved in Long et al. (2016). It is, however, hard in general to obtain precise information, in the same level of $\zeta_{\mathcal{Q}}(2)$, of the higher special values of $\zeta_{\mathcal{Q}}(n)$ ($n > 2$). Thus, we introduce the Apéry-like numbers $J_k(n)$ ($k = 0, 1, 2, \dots$) for each n defined through

¹Although the terminology “Apéry-like” is the identical, the usage/definition of the name in the current paper is different from the one in the title of Zagier (2009).

the *first anomaly* of $\zeta_Q(n)$ ($n > 2$) (Kimoto and Wakayama 2019) (see also Kimoto (2016)). These Apéry-like numbers share the properties of the one for $\zeta_Q(2)$, e.g. satisfy a similar recurrence relation as in the case of $\zeta_Q(2)$ and hence the ordinary differential equation satisfied by the generating function follows from the recurrence relation. Remarkably, the homogeneous part of each of the differential equations is identified with a (n dependent) power of the homogeneous part of the one corresponding to $\zeta_Q(2)$. Further, we observe that the meta-generating functions of Apéry-like numbers $J_k(n)$ are described explicitly by the modular Mahler measures studied by Rodriguez–Villegas in Rodriguez (1999). Through this relation, we may find an interesting aspect of a discrete dynamical system behind NCHO defined by a certain limit of finite abelian group via (*weighted*) *Cayley graphs* studied in Dasbach and Lalin (2009). Moreover, we note here (Kimoto and Wakayama 2012, 2019) that the generating function w_{2n} of Apéry-like numbers corresponding to the first anomaly in $\zeta_Q(2n)$ when $n = 2$ is given by an automorphic integral with a rational period function in the sense of Knopp (1978). This is obviously a generalization of our earlier result (Kimoto and Wakayama 2007) showing that w_2 is interpreted as a $\Gamma(2)$ -modular form of weight 1.

Furthermore, we show certain congruence relations among these normalized Apéry-like numbers which are the generalization of the results in Kimoto and Wakayama (2006). A possible generalization of the results in Liu (2018) seems very interesting. We also conjecture much stronger results based on numerical experiments in Kimoto and Wakayama (2019).

The Hamiltonian H_{Rabi} of the QRM is precisely given by

$$H_{\text{Rabi}} := \omega a^\dagger a + \Delta \sigma_z + g(a + a^\dagger) \sigma_x.$$

Here, $a^\dagger$ and a are the creation and annihilation operators of the single bosonic mode ($[a, a^\dagger] = 1$), σ_x, σ_z are the Pauli matrices (sometimes written as σ_1 and σ_3 , but since there is no risk of confusion with the variable x to appear below in the heat kernel, we use the usual notations), 2Δ is the energy difference between the two levels, and g denotes the coupling strength between the two-level system and the bosonic mode with frequency ω (subsequently, we set $\omega = 1$ without loss of generality). The integrability of the QRM was established in Braak (2011) using the well-known $\mathbb{Z}_2$ -symmetry of the Hamiltonian H_{Rabi} , usually called parity.

In the case of QRM, we recently obtained the (analytic formula of) heat kernel (Reyes and Wakayama 2019) using the Trotter–Kato product formula by extensive discussions of combinatorics and graph theory including quantum Fourier transform.

Concretely, the heat kernel $K_{\text{Rabi}}(t, x, y)$ of the QRM is given by

$$K_{\text{Rabi}}(t, x, y) = \tilde{K}_0(x, y, g, t) \sum_{\lambda=0}^{\infty} (t\Delta)^\lambda \Phi_\lambda(x, y, g, t).$$

Here the 2×2 matrix-valued function $\Phi_\lambda(g, t)$ for $\lambda \geq 0$ is given by

Fig. 1 From the NCHO to QRM (Heun's Pictures)

$$\Phi_\lambda(x, y, g, t) = \int \cdots \int_{0 \leq \mu_1 \leq \cdots \leq \mu_\lambda \leq 1} e^{\phi(\mu_\lambda, t) + \xi_\lambda(\mu_\lambda, t)} \begin{bmatrix} (-1)^\lambda \cosh & (-1)^{\lambda+1} \sinh \\ -\sinh & \cosh \end{bmatrix} \times (\theta_\lambda(x, y, \mu_\lambda, t)) d\mu_\lambda,$$

where $\mu_\lambda = (\mu_1, \mu_2, \dots, \mu_\lambda)$ and $d\mu_\lambda = d\mu_1 d\mu_2 \cdots d\mu_\lambda$ with $\mu_0 = 0$ and $d\mu_0 = 1$. For the definition of the functions $\phi, \xi_\lambda, \theta_\lambda$ and $\tilde{K}_0$, (Mehler's kernel) the reader is directed to Reyes and Wakayama (2019).

This is the first time an explicit determination of the heat kernel is obtained for an interacting system (though certain partial results have been discussed, e.g. in Legget 1987 for the Spin-Boson model and Anderson et al. 1970; Chakravarty 1995 for the Kondo effect using the Feynman–Kac formula.) The heat kernel formula allows us to have the contour integral representation of the spectral zeta function of the QRM (Sugiyama 2018) and open the study of the special values of negative integral points using it (Reyes and Wakayama 2019).

Further, although NCHO is not confirmed as a practical physical model, it may be considered as a “covering” model of QRM through the respective Heun ODE pictures (Wakayama 2016) (Fig. 1). Thus, in addition to the study of the respective number theoretical aspects of the models independently, the comparison of the number theoretic objects appearing from each model is an interesting and significant problem.

In addition to the number theoretic structure described above, we remark here that there appear certain algebraic curves, including elliptic and super elliptic curves, in the description of degenerations of the eigenstates for the asymmetric QRM with an integral perturbation parameter (Wakayama 2017; Kimoto et al. 2020; Reyes and Wakayama 2017). This shows another mathematical structure behind the asymmetric and symmetric QRM.

The following figure (Fig. 2) illustrates the position of this extended abstract from our whole interest. Particularly, the talk focused on the special values of such zeta

Fig. 2 Non-commutative harmonic oscillator and (asymmetric and symmetric) quantum Rabi models

functions (Ichinose and Wakayama 2005; Ochiai 2008; Kimoto and Wakayama 2006, 2007, 2012; Long et al. 2016; Liu 2018; Kimoto and Wakayama 2019). We note that special values of zetas may be considered as the moments of the partition function of the corresponding model.

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