The n th Root of NOT Operators of Quantum Computers

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Abstract This chapter proposes a novel approach to resolve the n th root of NOT problem for quantum computers using $(-1, 0, 1)$ permutation matrices. Only logic NOT and exchange operations are required. This result provides a complete solution to design and implement the n th root of NOT operators of quantum computers.

Keywords Quantum simulator · Quantum computation · Square root of NOT n -th root of NOT · Permutation matrix · Quantum logic gate

1 Introduction

Feynman [1] first proposed ‘universal quantum simulator’ towards a true quantum computer. Since then, research and development activities of quantum computation and quantum computers have become the new frontal of next-generation computers for two decades [2, 3]. Classical quantum mechanics use complex number vectors in Hilbert space to represent quantum states [4]. Any complex number is composed of two parts: a real part and an imaginary part. The imaginary number $i = \sqrt{-1}$ plays the essential role in the quantum mechanics construction. However, the mystery of the imaginary number causes severe difficulties for its manipulation, imagination and understanding [4–6]. Considering that modern computers are constructed by Boolean logic principles, how traditional logic structure is used to implement $\sqrt{-1}$ has been puzzling and deeply entangled in quantum computing for at least two decades [7–10]. Nothing in the published literature has described a way to implement this untamed operator using traditional logic operations [2, 11, 12].

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1.1 The Square Root of NOT Problem

Following traditional logic, negation corresponds to logic NOT ($\neg$). Initiated by Feynman [1] and further developed by Deutsch [9, 13], this problem has been represented as $\sqrt{\neg}$ ‘the Square Root of NOT’ as one of the most difficult issues in quantum computation especially in general quantum gates. They suggested resolving $\neg = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ equation using logic operations for the solution. Maglicki and Wang [11] provided an example of how to resolve the problem this way.

Let $\neg$ operation reverse two quantum spin states $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\neg|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle$$

$$\neg|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle$$

To apply unitary rotational matrices, $\sqrt{\neg}$ operator can be expressed as

$$\sqrt{\neg} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

In the equations, both $e^{i\pi}$ and i symbols are involved. From a representative viewpoint, equations are useless because the symbols i and $\sqrt{\neg}$ are both logic equivalent. The equations are in circular definitions.

To explore how to use traditional logic implementing $\sqrt{\neg}$, it is necessary to analyse what has been established at the foundation levels of modern complex number construction.

1.2 Complex Number in History

The origin and development of complex number has a long and mysterious history [14–16]. In the nineteenth century, Gauss and Euler [15] made their foundation contributions to formally identifying imaginary parts as the most essential components to resolve solutions from n th algebraic equations. After their work, the imaginary number has been gradually accepted by mainstream mathematicians to be one of the most important parts of mathematics [15]. Hamilton established consistent operations on complex number in 1837 [17]. He constructed a complex number $a + bi$ as an ordered number pair (a, b) .

For example, let $a + bi$ and $c + di$ be two complex numbers. Four essential operations: $\{\pm, \bullet, /\}$ can be expressed as

$$\begin{aligned}
(a, b) \pm (c, d) &= (a \pm c, b \pm d) \\
(a, b) \bullet (c, d) &= (ac - bd, ad + bc) \\
\frac{(a, b)}{(c, d)} &= \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2} \right)
\end{aligned}$$

Using ordered pair representation, complex number operations are firmly established on real number operations. No further mysterious characteristics of imaginary numbers remain in the equations because all operations are well defined in real number construction.

2 Solution of the Square Root of NOT Problem

If we apply an imaginary number to an ordered pair, we have

$$i : (a, b) \rightarrow (-b, a)$$

When we do not restrict $\sqrt{-1}$ solution in $\{0, 1\}$ field but extend the field to $\{-1, 0, 1\}$. A permutation matrix can be constructed.

Let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_2^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I_2^- = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z_2^\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$Z_2 : (a, b) \rightarrow (-b, a)$$

$$(-b, a) = (a, b) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Because Z_2 provides the same result as the imaginary number when applied to the pair, it is necessary for us to explore Z_2 features in details.

Two eigenvalues of Z_2 can be determined from its determinant.

$$\begin{aligned}
|\lambda I_2 - Z_2| &= \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda \end{vmatrix} = 0 \\
\lambda^2 + 1 &= 0, \quad \lambda^2 = -1, \quad \lambda = \pm\sqrt{-1}
\end{aligned}$$

This corresponds to either $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ as the solution. There are two unitary matrices U_+ , U_- and two Hermite conjugate matrices U_+^* , U_-^* undertaken similarity transformation on Z_2 to produce the two diagonal matrices:

$$iI_2^\pm = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = U_+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_+^*,$$

$$iI_2^\mp = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = U_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_-^*$$

Although three matrices belong to one matrix group under similarity transformation, five matrices can be distinguished without any direct equality.

$$iI_2 \neq iI_2^\pm \neq Z_2 \neq iI_2^\mp \neq -iI_2$$

To apply the five matrices twice separately, they all equal to $-I_2$.

$$(\pm iI_2)^2 = \begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix} \begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

$$(iI_2^\pm)^2 = (iI_2^\mp)^2 = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix} \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

and

$$Z_2^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

Therefore, the Z_2 matrix is an equivalent form of the imaginary number under the transformation.

For any ordered pair (a, b) ,

$$(Z_2)^2 : (a, b) \rightarrow (-a, -b)$$

$$(Z_2)^2 : (a, b) \xrightarrow{Z_2} (-b, a) \xrightarrow{Z_2} (-a, -b)$$

$$(Z_2)^2 = -I_2$$

$$Z_2 = \sqrt{-I_2}$$

So, $\sqrt{-}$ operation can be constructed originally from one-one correspondences from the Z_2 matrix.

Let $\langle x|$ be a quantum state, $\neg\langle x| = \langle \bar{x}|$. For a non-zero element of Z_2 , two values $\{-1, 1\}$ of the elements map $\begin{cases} -1 : \langle x| \xrightarrow{\neg} \langle \bar{x}| \\ 1 : \langle x| \rightarrow \langle x| \end{cases}$ then a $\sqrt{-}$ operator is generated from a Z_2 operator.

For an ordered state pair $(\langle x|, \langle y|)$,

$$(\langle x|, \langle y|) \xrightarrow{\sqrt{-}} (\langle \bar{y}|, \langle x|) \xrightarrow{\sqrt{-}} (\langle \bar{x}|, \langle \bar{y}|) = \neg(\langle x|, \langle y|)$$

Therefore, Z_2 is a homologous form of the $\sqrt{-}$ operator.

Under this construction, the square root of NOT problem in quantum computation is solved entirely. Only two elementary operations are involved in the transformation: logic **NOT** operation and pair-state exchange, respectively. They can be implemented readily using traditional logic constructions.

3 General Solution of the n th Root of NOT Operation

In this part, a general solution of $\sqrt[n]{-}$ ‘the n th root of NOT’ for quantum computers is explored.

Let J_n denote a conjugate permutation matrix which contains n columns and n rows and each row (column) has one non-zero element.

$$J_n = (J_{i,j}), \quad 1 = \sum_{i=1}^n |J_{i,j}| = \sum_{i=1}^n |J_{i,j}|, \quad J_{i,j} \in \{-1, 0, 1\}, \quad i, j \in [1, n]$$

Let I_n be a unit matrix, $I_{i,j} = 1, i = j; I_{i,j} = 0, i \neq j, i, j \in [1, n]$.

For example, matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

are J_n matrices.

Let P_n be a $(0, 1)$ -permutation matrix in which each column (row) contains only one element, and $PS(n)$ denote a permutation space containing all P_n matrices.

Let $JS(n)$ denote a conjugate permutation space.

Lemma For a given n , $PS(n)$ contains a total number of $n!$ distinguishable matrices, that is, $|PS(n)| = n!$.

Theorem For a given n , $JS(n)$ contains a total number of $2^n n!$ distinguishable matrices, $|JS(n)| = 2^n n!$.

Proof Each non-zero element of J_n has two values $\{-1, 1\}$, and n different elements have 2^n selections. The n elements can select a total number of $n!$ different positions. Both symbol and position selections are independent, and each combination determines a J_n matrix. So there are $2^n n!$ distinguishable matrices.

Corollary $JS(n)$ is a matrix space that is 2^n times larger than $PS(n)$.

Theorem A matrix group of simple rotation in $JS(n)$ may contain $2n$ distinguishable matrices.

Proof Using a rotation matrix $Z_n \in JS(n)$,

$$Z_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & & \\ \dots & & & \dots & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad J_{i,i+1} = 1, i \in [1, n], J_{n,1} = -1 \text{ and a vector}$$

$$X = (1 \ 2 \ 3 \ \dots \ n-1 \ n).$$

To apply $2n$ Z_n matrices sequentially to the vector X , the following $2n$ vectors are produced:

$$\begin{pmatrix} X = XZ_n^{2n} \\ XZ_n \\ \dots \\ XZ_n^n \\ XZ_n^{n+1} \\ \dots \\ XZ_n^{2n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ -n & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & -n+2 & -n+1 & -n \\ n & -1 & -2 & \dots & -n+3 & -n+2 & -n+1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 3 & 4 & \dots & n-1 & n & -1 \end{pmatrix}.$$

That is, $2n$ distinguishable matrices $\{Z_n^j\}_{j=1}^{2n}$, $Z_n^0 = Z_n^{2n} = I_n$ are included.

Because of $X \xrightarrow{Z_n^n} -X \xrightarrow{Z_n^n} X$, there are $Z_n^n = -I_n$ and $Z_n^{2n} = I_n$, that is, $Z_n^n = -I_n$.

Theorem For a Z_n , there are n eigenvalues $\{\lambda_i\}_{i=1}^n$, $\lambda_i = \sqrt[n]{-1}$, $i \in [1, n]$.

Proof

$$|\lambda I_n - Z_n| = \begin{vmatrix} \lambda - 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & -1 \\ 1 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix} = \lambda^n + 1 = 0.$$

Therefore, $Z_n = \sqrt[n]{-1} I_n$.

For non-zero values, $\begin{cases} 1 : \langle x | \rightarrow \langle x | \\ -1 : \langle x | \rightarrow \langle \bar{x} | \end{cases} \text{ map } Z_n \rightarrow \sqrt[n]{-1}.$

Theorem For any state vector X , $X \left(\sqrt[n]{\neg} \right)^n = \neg X$.

Proof

$$\begin{pmatrix} X \\ X \sqrt[n]{\neg} \\ X \sqrt[n]{\neg}^{n-1} \\ X \sqrt[n]{\neg}^n = \neg X \end{pmatrix} = \begin{pmatrix} \langle 1 | \langle 2 | \langle 3 | \dots \langle n | \\ \langle \bar{n} | \langle 1 | \langle 2 | \dots \langle n-1 | \\ \dots \dots \dots \dots \dots \\ \langle \bar{2} | \langle \bar{3} | \langle \bar{4} | \dots \langle 1 | \\ \langle \bar{1} | \langle \bar{2} | \langle \bar{3} | \dots \langle \bar{n} | \end{pmatrix}.$$

4 Conclusion

Using $(-1,0,1)$ permutation matrices as basic tools, the n th root of NOT operators for quantum computers can be constructed and implemented by the traditional logic structure. Considering that this problem has puzzled advanced research of quantum computer for 20 years, this solution can provide quantum computer designers to practically implement quantum computers using traditional logic. The details of this construction will investigate in other places and the relationships among conjugate logic, quantum logic, quantum gates and complex number structures will be explored for foundation of Quantum computers and quantum computation of future computers.

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