Acta Applicandae Mathematicae **27**: 47–54, 1992.

Baillon's Theorem on Maximal Regularity

B. EBERHARDT and G. GREINER
Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 7400 Tübingen, Germany

(Received: 27 April 1992)

Abstract. The aim of this note is to give a proof of *Baillon's Theorem on Maximal Regularity*. Though it is in some sense a negative result (it states that for abstract Cauchy problems maximal regularity can occur only in very special cases), it is commonly accepted that it is important. Many people believe that its proof is very complicated. This might be due to the fact that Baillon's note in the *Comptes Rendus* is rather short and sometimes difficult to understand. The proof outlined here follows basically Baillon's lines. However it is simplified and (hopefully) easier to understand.

Mathematics Subject Classifications (1991): 47D03, 34A12

Key words: Baillon's theorem, abstract Cauchy problem, strongly continuous semigroup

A strongly continuous semigroup $\mathbf{T} = (T(t))_{t\geq 0}$ on a Banach space X is said to have the *maximal regularity property* **(MR)** if the following condition is satisfied:

For every $f \in C([0,\tau], X)$ the convolution $\mathbf{T} * f$ which is defined by $(\mathbf{T} * f)(t) := \int_0^t T(t-s)f(s)\,ds$ is continuously differentiable.

It is not difficult to verify that whenever **(MR)** is true for some $\tau > 0$, then it is true for every τ. Thus we can assume without loss of generality that $\tau = 1$. From the identity

$$\frac{1}{h}((\mathbf{T} * f)(t+h) - (\mathbf{T} * f)(t))$$
$$= \frac{1}{h}(T(h) - Id)(\mathbf{T} * f)(t) + \frac{1}{h}\int_0^h T(s)f(t+h-s)\,ds$$

it follows that for a continuous f we have $\mathbf{T} * f \in C^1([0,1], X)$ if and only if $\mathbf{T} * f \in C([0,1], X_1)$ where X_1 is the Banach space $D(A)$ equipped with the graph norm.[1] Thus **(MR)** can be restated as follows:

[1] Note that a function which is differentiable from the right and has a continuous right side derivate is actually C^1.

For every $f \in C([0,\tau],X)$ the convolution $\mathbf{T} * f$ is a continuous function from $[0,\tau]$ to X_1.

The convolution $\mathbf{T} * f$ is of interest, since it is the only possible solution of the inhomogeneous Cauchy problem **(CP)**

$$\dot{u}(t) = Au(t) + f(t), \qquad u(0) = 0.$$

More precisely, whenever $u \in C^1([0,1],X) \cap C([0,1],D(A))$ satisfies **(CP)** then $u = \mathbf{T} * f$.

Trivial examples of semigroups satisfying **(MR)** are those with a bounded generator. Then $t \mapsto T(t)$ is C^∞ on the interval $[0,\tau]$ which implies that $\mathbf{T} * f$ is C^∞ for every $f \in C([0,\tau],X)$. An example of an unbounded generator A satisfying **(MR)** is the multiplication operator on c_0, the space of all null sequences, defined by $A(\xi_n) := (-n \cdot \xi_n)$. The corresponding semigroup is given by $T(t)x = (e^{-nt}x_n)_{n\in\mathbf{N}}$ for $x = (x_n)_{n\in\mathbf{N}} \in c_0$. Given $f \in C([0,1],c_0)$ then $f = (f_n)$ with $f_n \in C[0,1]$ and $\lim_{n\to\infty} \|f_n\| = 0$. A straightforward calculation shows that $\mathbf{T} * f =: g = (g_n)$, where $g_n \in C[0,1]$ is given by $g_n(t) := \int_0^t e^{-n(t-s)} f_n(s)\, ds$. It follows easily that $g(t) \in D(A)$ for all $t \geq 0$ and that $g\colon [0,1] \to D(A)$ is continuous at every $t > 0$. In order to show continuity at $t = 0$ we apply the (second) mean value theorem on integrals and obtain

$$-ng_n(t) = -n \int_0^t e^{-n(t-s)}\, ds \cdot f(\xi_{n,t}) = -(1 - e^{-nt}) \cdot f(\xi_{n,t})$$

for suitable $\xi_{n,t} \in [0,t]$. Then given $\varepsilon > 0$, $|-ng_n(t)| \leq 1 \cdot \|f_n\| < \varepsilon$ for $n \geq N = N(\varepsilon)$ (uniformly in $t \in [0,1]$). Moreover, for $n < N = N(\varepsilon)$ we have $|-ng_n(t)| \leq (1 - e^{-Nt})\|f_n\| < \varepsilon$ for t sufficiently small. We conclude that $\|Ag(t)\| \to 0$ as $t \to 0$, hence $g\colon [0,1] \to D(A)$ is continuous at 0 as well.

In the example mentioned above the choice of the space c_0 was crucial! As a consequence of Baillon's Theorem this cannot be true in L^p-spaces. In fact, the result states that unbounded generators which have **(MR)** can only exist in Banach spaces containing a closed subspace which is isomorphic to c_0. We need the following characterization of Banach spaces containing c_0.

THEOREM 0.1 *A Banach space X contains a closed subspace which is isomorphic to c_0 if and only if there exist a sequence $(x_n) \subset X$ and a constant M such that*

$$\inf_{n\in\mathbf{N}} \{\|x_n\|\} > 0 \qquad \text{and} \qquad \|x_0 \pm x_1 \pm x_2 \pm \ldots \pm x_n\| \leq M$$

for every $n \in \mathbf{N}$ and all possible choices of signs $+$ or $-$. (1)

The proof follows from joint work of C. Bessaga and A. Pelczyński [2, Coroll. 1 and Lemma 3]. We sketch a direct proof in the appendix.

Another ingredient for the proof is the following result of Hille (cf. [5] or [6, 2.5.3]).

THEOREM 0.2 *Let* $(T(t)$ *be a* C_0*-semigroup on* X *with generator* A*. If for every* $x \in X$ *the mapping* $t \mapsto T(t)x$ *is differentiable on* $(0, \infty)$ *and* $\limsup_{t\to 0} t \, \|AT(t)\| < \frac{1}{e}$ *then* A *is a bounded operator.*

COROLLARY 0.3 *If* $(T(t))$ *is a semigroup with an unbounded generator* A *satisfying* **(MR)** *then* $\operatorname{Im} T(t) \subset D(A)$ *for* $t > 0$ *and* $\limsup_{t\to 0} t\|AT(t)\| \geq \frac{1}{e}$.

Proof. For $x \in X$ we consider the function $f(t) := T(t)x$. Then $(\mathbf{T} * f)(t) = t \cdot T(t)x$. By **(MR)** this function is C^1, hence $t \mapsto T(t)x$ is C^1 on $(0, \infty)$. It follows that $\operatorname{Im} T(t) \subset D(A)$ for $t > 0$. Moreover, the theorem implies $\limsup_{t\to 0} t\|AT(t)\| \geq \frac{1}{e}$. Q.E.D.

A function $f\colon [0, 1] \to X$ is said to be *piecewise continuous* if f is continuous except at finitely many points $0 < t_1 < \ldots < t_n < 1$ and such that right- and lefthand limits exist at every point t_i. The set of all piecewise continuous functions will be denoted by $C_{pw}([0, 1], X)$.

We will show that for a semigroup $\mathbf{T}$ satisfying **(MR)** and a piecewise continuous f the convolution $\mathbf{T} * f$ is a continuous mapping into $X_1 := (D(A), \|\cdot\|_A)$.

PROPOSITION 0.4 *If* $\mathbf{T}$ *satisfies* **(MR)** *and* $f \in C_{pw}([0, 1], X)$*, then* $\mathbf{T} * f \in C([0, 1], X_1)$*. Moreover, there is a constant* C *such that*

$$\sup_{0\leq t\leq 1} \|A(\mathbf{T} * f)(t)\| \leq C \cdot \sup_{0\leq t\leq 1} \|f(t)\| \qquad \textit{for all} \quad f \in C_{pw}([0, 1], X)(2).$$

Proof. We only consider the case where f has one discontinuity at t_1 say. The functions $t \mapsto f(t)$ and $t \mapsto f(t_1 + t)$ defined on $[0, t_1)$ and $(0, 1 - t_1]$ have continuous extensions f_0 and f_1 say. Then

$$(\mathbf{T}*f)(t) = \begin{cases} (\mathbf{T} * f_0)(t) & \text{if } t \in [0, t_1], \\ (T(t - t_1)(\mathbf{T} * f_0)(t_1)) + (\mathbf{T} * f_1)(t - t_1) & \text{if } t \in (t_1, 1]. \end{cases} \quad (3)$$

It follows that both $\mathbf{T}*f$ and $A(\mathbf{T}*f)$ are continuous. Thus $\mathbf{T}*f \in C([0, 1], X_1)$.

First we observe that the mapping $f \mapsto \mathbf{T} * f$ is continuous from $(C_{pw}([0, 1], X), \|\cdot\|_\infty)$ into $(C([0, 1], X), \|\cdot\|_\infty)$. In fact this follows from the estimate

$$\|(\mathbf{T} * f)(t)\| \leq \int_0^t \|T(t - s)\| \, \|f(s)\| \, ds \leq \left(\int_0^1 \|T(s)\| \, ds\right) \|f\|_\infty.$$

The considerations above show that the range of this mapping is contained in the Banach space $C([0, 1], X_1)$ which is continuously embedded in $C([0, 1], X)$. Thus by the closed graph theorem $f \mapsto \mathbf{T} * f$ is continuous from $C_{pw}([0, 1], X)$ into $C([0, 1], X_1)$. It follows that there is a constant C such that (1) holds. Q.E.D.

Now we have all the prerequisites in order to prove the main result.

THEOREM 0.5 (Baillon's Theorem) *Let* A *be the generator of a* C_0*-semigroup* $(T(t))$ *on a Banach space* X *satisfying* **(MR)**. *Then either* A *is bounded or* X *contains a closed subspace which is isomorphic to* c_0.

Proof. We assume that $(A, D(A))$ is an unbounded generator of a C_0-semigroup $(T(t))$ on a Banach space X satisfying **(MR)**. In order to prove the Theorem we have to show that X contains a sequence (x_n) with the properties stated in Theorem 0.1. Q.E.D.

Construction of the Sequence. Because of the Corollary to Theorem 0.2 we can find a sequence of positive real numbers $(t_i)_{i\in\mathbf{N}}$ such that

$$t_0 := 1, \quad t_i < \frac{1}{2^i} t_{i-1} \quad \text{for every } i = 1, 2, 3, \ldots, \tag{4}$$

and

$$\|t_i A T(t_i)\|_{\mathcal{L}(X)} > \frac{1}{2e} \qquad \text{for every} \quad i = 1, 2, 3, \ldots. \tag{5}$$

Then there are elements $y_i \in X$, $\|y_i\|_X \leq 1$ such that $\|t_i A T(t_i) y_i\|_X > \frac{1}{2e}$ for $i = 1, 2, 3, \ldots$ and $AT(1)y_0 \neq 0$. From Proposition 0.4 we deduce that for all i

$$\|t_i A T(t_i) y_i\|_X = \|A(\mathbf{T} * \tilde{y}_i)(t_i)\|_X \leq C \cdot \|\tilde{y}_i\| \leq \tilde{C}, \tag{6}$$

where $\tilde{y}_i(t) := T(t) y_i$, $\tilde{C} := C \cdot \sup_{0 \leq t \leq 1} \|T(t)\|$.

If we define:

$$x_i = t_i A T(t_i) y_i \qquad \text{for} \quad i = 0, 1, 2, \ldots, \tag{7}$$

we have at once

$$\inf_{i \geq 0} \|x_i\| > 0. \tag{8}$$

It remains to verify the second condition of (1). We therefore choose $n \in \mathbf{N}$ and ε_i with $\varepsilon_i = \pm 1$ for $i = 0, 1, \ldots n$. Defining the following piecewise continuous function:

$$f(s) := \begin{cases} \varepsilon_i T(s + t_i - 1) y_i, & 1 - t_i \leq s \leq 1 - t_{i+1}, \\ & i = 0, \ldots, n-1, \\ \varepsilon_n T(s + t_n - 1) y_n, & 1 - t_n \leq s \leq 1, \end{cases}$$

we obtain

$$\begin{aligned} (\mathbf{T} * f)(1) &= \int_0^1 T(1-s) f(s)\, ds \\ &= \sum_{i=0}^{n-1} \varepsilon_i \int_{1-t_i}^{1-t_{i+1}} T(1-s) T(s + t_i - 1) y_i\, ds \\ &\quad + \varepsilon_n \int_{1-t_n}^{1} T(1-s) T(s + t_n - 1) y_n\, ds \\ &= \sum_{i=0}^{n-1} \varepsilon_i (t_i - t_{i+1}) T(t_i) y_i + \varepsilon_n t_n T(t_n) y_n. \end{aligned}$$

Therefore we get using (6), (7) and the triangle inequality:

$$\begin{aligned}
&\|\varepsilon_0 x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n - A(\mathbf{T} * f)(1)\|_X \\
&\quad \leq \|\varepsilon_0 t_1 AT(1) y_0 + \varepsilon_1 t_2 AT(t_1) y_1 \\
&\qquad + \ldots + \varepsilon_{n-1} t_n AT(t_{n-1}) y_{n-1} + 0\|_X \\
&\quad = \left\| \varepsilon_0 t_1 x_0 + \varepsilon_1 \frac{t_2}{t_1} x_1 + \ldots + \varepsilon_{n-1} \frac{t_n}{t_{n-1}} x_{n-1} \right\|_X \\
&\quad \leq \frac{1}{2}\tilde{C} + \frac{1}{4}\tilde{C} + \ldots + \frac{1}{2^n}\tilde{C} \leq \tilde{C}.
\end{aligned}$$

Together with the inequality of Proposition 0.4,

$$\|A(\mathbf{T} * f)(1)\| \leq C\|f\| \leq C \cdot \sup_{0 \leq t \leq 1} \|T(t)\|$$

we obtain

$$\begin{aligned}
&\|\varepsilon_0 x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|_X \\
&\quad \leq \|\varepsilon_0 x_0 + \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n \\
&\qquad - A(\mathbf{T} * f)(1)\|_X + \|A(\mathbf{T} * f)(1)\| \\
&\quad \leq C + C \cdot \sup_{0 \leq t \leq 1} \|T(t)\| < \infty,
\end{aligned}$$

independently of $n \in \mathbf{N}$ and the choice of $\varepsilon_i = \pm 1$. Q.E.D.

A few words to the consequences of Baillon's Theorem. Recall that closed subspaces of reflexive Banach spaces are also reflexive. Therefore on reflexive Banach spaces all semigroups satisfying **(MR)** are uniformly continuous (and therefore not very interesting). More generally, from the fact that c_0 is not weakly sequentially complete (the sequence $(\sum_{i=1}^{n} e_i)_{n \in \mathbf{N}}$ (e_i the i-th unit vector) is weak Cauchy but not weakly convergent), it follows that on weakly sequentially complete spaces **(MR)** can occur only when the generator is bounded. In addition to reflexive spaces the L^1-spaces are weakly sequentially complete ([4, IV.8.6]). The Sobolev spaces $W^{p,k}$ can be considered as closed subspaces of products of L^p-spaces. Hence they are weakly sequentially complete as well.

For example the semigroup generated by the multiplication operator $A(\xi_n) = (-n \cdot \xi_n)$ is maximal regular on c_0 but *not* on ℓ^p, $1 \leq p < \infty$.

Or the other way around, if you have an unbounded generator on an L^p-space $(1 \leq p < \infty)$, a Sobolev space or a reflexive Banach space (e.g. Hilbert space), then there will be always a continuous function $f \in C([0, \tau], X)$ such that the inhomogeneous Cauchy problem

$$\frac{du}{dt} = Au + f,$$

$$u(0) = 0$$

has *no* classical solution.

Appendix

In this Appendix we want to give a direct proof of Theorem 0.1. It is based on the following lemma on infinite matrices $A = (a_{ij})_{i,j\in\mathbf{N}}$. A *submatrix* of A is an infinite matrix $B = (b_{ij})_{i,j\in\mathbf{N}}$ obtained from A by cancelling some rows and the corresponding columns. In other words, there is a subsequence $(n_j)_{j\in\mathbf{N}}$ of the natural numbers such that $b_{ij} = a_{n_i n_j}$. In the following we consider matrices which give rise to bounded linear operators on ℓ^1. Thus

all columns are elements of ℓ^1 and
the ℓ^1-norm of the columns is uniformly bounded $(\star)$

The norm of the induced operator is the supremum of the ℓ^1-norm of the columns of A.

LEMMA 0.6 *Let A be an infinite matrix satisfying $(\star)$. If the diagonal (a_{ii}) does not converge to zero, then there is a submatrix B of A which induces an isomorphism on ℓ^1.*

Proof. By assumption there is a subsequence (n_i) and $\delta > 0$ such that $\inf_{i\in\mathbf{N}}\{|a_{n_i n_i}|\} \geq \delta > 0$. Thus considering the submatrix defined by (n_i) we can assume w.l.o.g. that $\inf_{i\in\mathbf{N}}\{|a_{ii}|\} \geq \delta > 0$.

Now we show that for every $\varepsilon > 0$ there is a submatrix B of A which satisfies $\sum_{i\neq j}|b_{ij}| < \varepsilon$ for every $j \in \mathbf{N}$. It follows that $\|B - D\| \leq \varepsilon$ where D denotes the diagonal part of B. In case $\varepsilon < \delta$ the matrix B is invertible, because its diagonal part is invertible with $\|D^{-1}\| \leq \delta^{-1}$ and $\|B - D\| \leq \varepsilon < \delta$.

We construct B in two steps.

1) There is a submatrix C such that $\sum_{i>j}|c_{ij}| < \frac{\varepsilon}{2}$ for every j.
2) There is a submatrix B of C such that $\sum_{i<j}|b_{ij}| < \frac{\varepsilon}{2}$ for every j.

Step 1)

Define $n_1 := 1$. Since the first column is ℓ^1 there is a subsequence $(n_1, n_{12}, n_{13}, \ldots)$ of $\mathbf{N}$ such that $\sum_{i=2}^{\infty}|a_{n_{1i}1}| < \frac{\varepsilon}{2}$. Define $n_2 := n_{12}$. Since the n_2^{th} column is ℓ^1 there is a subsequence $(n_1, n_2, n_{23}, n_{24}, \ldots)$ of $(n_1, n_{12}, n_{13}, \ldots)$ such that $\sum_{i=3}^{\infty}|a_{n_{2i}n_2}| < \frac{\varepsilon}{2}$. Define $n_3 := n_{23}$. Proceeding this way one obtains (recursively) a subsequence such that the corresponding submatrix C satisfies $\sum_{i>j}|c_{ij}| < \frac{\varepsilon}{2}$ for every j.

Step 2)

Let c be a bound for the ℓ^1-norm of the columns. If we choose $m \in \mathbf{N}$ such that $m \cdot \frac{\varepsilon}{4} > c$, then among the first m rows of B there must be one which contains infinitely many elements of absolute value less that $\frac{\varepsilon}{4}$. (Otherwise there are columns which have ℓ^1-norm greater than $m \cdot \frac{\varepsilon}{4} > c$ which is a contradiction).

Let n_1 be such a row and choose a subsequence $(n_1, n_{12}, n_{13}, \ldots)$ of $(n_1, m+1, m+2, , \ldots)$ such that $|b_{n_1,n_{1j}}| < \frac{\varepsilon}{4}$ for $j \geq 2$.

For the same reason as above there is among the $2m$ rows $n_{12}, n_{13}, \ldots, n_{1,2m+1}$ one which has infinitely many elements of absolute value less than $\frac{\varepsilon}{8}$. Let n_2 be such a row and choose a subsequence $(n_1, n_2, n_{23}, n_{24}, \ldots)$ of $(n_1, n_2, n_{1,2m+2}, \ldots)$ such that $|b_{n_2,n_{2j}}| < \frac{\varepsilon}{8}$ for all $j \geq 3$. Proceeding this way one obtains (recursively) a subsequence such that the corresponding submatrix B of C satisfies $|b_{ij}| < 2^{-i-1}\varepsilon$ for every $j > i$. Hence $\sum_{i<j} |b_{ij}| < \sum_{i=1}^{j-1} 2^{-i-1}\varepsilon < \frac{\varepsilon}{2}$ for every j. Q.E.D.

Now we can give the

Proof of Theorem 0.1. Let (x_n) be sequence in the Banach space X such that $\delta := \inf_{n\in\mathbf{N}} \|x_n\| > 0$ and $\|x_0 \pm x_1 \pm x_2 \pm \ldots \pm x_n\| \leq M$ for every $n \in \mathbf{N}$ and all possible choices of signs. By the Hahn–Banach theorem there exist linear functionals $x'_n \in X'$ such that

$$\|x'_n\| = 1 \qquad \text{and} \qquad < x_n, x'_n >= \|x_n\| \geq \delta \qquad \text{for all } n \in \mathbf{N}.$$

The infinite matrix $A = (< x_i, x'_j >)_{ij\in\mathbf{N}}$ satisfies the hypotheses of the Lemma. In fact, $|a_{jj}| = \|x_j\| \geq \delta$ and $\sum_{i=0}^{n} |a_{ij}| = \sum_{i=0}^{n} | < x_i, x'_j > | = \langle \sum_{i=0}^{n} \varepsilon_i x_i, x'_j \rangle$ where $\varepsilon_i = \operatorname{sgn}(< x_i, x'_j >)$. It follows that $\sum_{i=0}^{n} |a_{ij}| \leq \| \sum_{i=0}^{n} \varepsilon_i x_i\| \cdot \|x'_j\| \leq M$ for every $n, j \in \mathbf{N}$.

According to the Lemma we choose a subsequence (n_i) which defines an invertible submatrix B of A. Let $y_i := x_{n_i}$, $y'_i := x'_{n_i}$.

We define a linear mapping T_0 from the space of all finite sequences φ into X by $T_0(\xi_n) := \sum_n \xi_n y_n$. We claim that T_0 is bounded. In fact, each $\sum_n \xi_n y_n$ with $(\xi_n) \in \varphi$, $\|(\xi_n)\| \leq 1$ is a convex combination of vectors $\pm x_0 \pm x_1 \pm x_2 \pm \ldots x_m$. Since each of these vectors has norm less than M so has $\sum_n \xi_n y_n$. It follows that T_0 is bounded and has norm $\leq M$. T_0 can be uniquely extended to a bounded linear map $T: c_0 \to X$.

Furthermore we define $S: \ell^1 \to X'$ by $S(\eta_n) := \sum_n \eta_n y'_n$. Obviously S is linear and bounded ($\|S\| \leq 1$) and it is easily verified that the composition $T' \circ S: \ell^1 \to X' \to \ell^1$ is represented by the matrix B. Thus $T' \circ S$ is invertible and therefore its adjoint $S' \circ T''$ as well. For $\xi \in c_0 \subset \ell^\infty$ we have $\|T\xi\| = \|T''\xi\| \geq \|S'\|^{-1}\|S' \circ T''\xi\| \geq \|S'\|^{-1}\|(S' \circ T'')^{-1}\|^{-1}\|\xi\|$. This shows that T is an isomorphism of c_0 onto a subspace of X. Q.E.D.

References

1. J.B. Baillon, Caractère borné de certains générateurs de semigroupes linéaires dans les espaces de Banach, *C.R. Acad. Sc. Paris* **290** (1980), pp. 757–760.
2. C. Bessaga, A. Pelczyński, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **XVII** (1958), pp. 329–396.
3. Ph. Clément (et al.), *One-Parameter Semigroups*, CWI Monographs **5**, North-Holland, Amsterdam, 1987.
4. N. Dunford, J.T. Schwartz, *Linear Operators, I*, Wiley, New York, 1958.

5. E. Hille, On the differentiability of semigroups of operators, *Acta Sc. Math. (Szeged)* **12** (1950), pp. 19–24.
6. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.