Planar Octilinear Drawings with One Bend Per Edge[★]

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Abstract. In *octilinear drawings* of planar graphs, every edge is drawn as an alternating sequence of horizontal, vertical and diagonal line-segments. In this paper, we study octilinear drawings of low edge complexity, i.e., with few bends per edge. A k -planar graph is a planar graph in which each vertex has degree less or equal to k . In particular, we prove that every 4-planar graph admits a planar octilinear drawing with at most one bend per edge on an integer grid of size $O(n^2) \times O(n)$. For 5-planar graphs, we prove that one bend per edge still suffices in order to construct planar octilinear drawings, but in super-polynomial area. However, for 6-planar graphs we give a class of graphs whose planar octilinear drawings require at least two bends per edge.

1 Motivation

Drawing edges as octilinear paths plays a central role in the design of metro-maps (see e.g., [9,18,19]), which dates back to the 1930's when Henry Beck, an engineering draftsman, designed the first schematic map of London Underground using mostly horizontal, vertical and diagonal segments. Laying out networks in such a way is called *octilinear graph drawing*, i.e., an *octilinear drawing* of a (planar) graph $G = (V, E)$ of maximum degree eight is a (planar) drawing $\Gamma(G)$ of G in which each vertex occupies a point on the integer grid and each edge is drawn as a sequence of horizontal, vertical and diagonal line-segments.

For drawings of (planar) graphs to be readable, special care is needed to keep the number of bends small. However, the problem of determining whether a given embedded 8-planar graph (that is, a planar graph of maximum degree eight with given combinatorial embedding) admits a bendless octilinear drawing is NP-hard [17]. This negative result motivated us to study octilinear drawings of *low edge complexity*, that is, with few bends per edge. Surprisingly enough, very few results relevant to this problem were known, even if the octilinear model has been well-studied in the context of metro-map visualization and map schematization (see e.g. [21]). As an immediate byproduct

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of a result of Keszegh et al. [13], it turns out that every d -planar graph, with $3 \leq d \leq 8$, admits a planar octilinear drawing with at most two bends per edge; see Section 1. On the other hand, every 3-planar graph on five or more vertices admits a planar octilinear drawing in which all edges are bendless [6,12].

In this paper, we bridge the gap between the two aforementioned results. We prove that every 4-planar graph admits a planar octilinear drawing with at most one bend per edge in cubic area. We also show that every 5-planar graph also admits a planar octilinear drawing with at most one bend per edge, but our construction may require super-polynomial area. Finally, we demonstrate an infinite class of 6-planar graphs whose planar octilinear drawings require at least two bends per edge.

Related Work. The research on the (planar) *slope number of graphs* focuses on minimizing the number of used slopes (see e.g., [10,13,14,15,16]). Octilinear drawings can be seen as a special case thereof, since only four slopes are used. Keszegh et al. [13] showed that any d -planar graph admits a planar drawing with one bend per edge, in which all edge-segments have at most $2d$ different slopes. So, for $d = 4$ and $d = 5$, we reduce the number of different slopes from 8 and 10 to 4. They also proved that d -planar graphs, $d \geq 3$, admit planar drawings with two bends per edge that require at most $\lceil \frac{d}{2} \rceil$ different slopes. One can transfer this technique to the octilinear model and show that any d -planar graph, with $3 \leq d \leq 8$, admits a planar octilinear drawing with two bends per edge. For $d = 3$, Di Giacomo et al. [6] recently proved that any 3-planar graph with $n \geq 5$ vertices has a bendless planar drawing with at most 4 different slopes and angular resolution $\pi/4$ (see also [12]); their approach also yields octilinear drawings.

Tamassia [20] showed that one can minimize the total number of bends in orthogonal drawings of embedded 4-planar graphs. However, minimizing the number of bends over all embeddings of a 4-planar graph is NP-hard [7]. The core of Tamassia's approach is a min-cost flow algorithm that specifies the angles and the bends of the drawing, producing an *orthogonal representation*, and then computes the actual drawing by specifying the drawing's exact coordinates. Tamassia's algorithm can be employed to produce a bend-minimum octilinear representation for any given embedded 8-planar graph. However, a bend-minimum octilinear representation may not be realizable by a corresponding planar octilinear drawing [4].

Biedl and Kant [2] showed that any 4-planar graph except the octahedron admits a planar orthogonal drawing with at most two bends per edge on an $O(n^2)$ integer grid. Hence, the octilinear drawing model allows us to reduce the number of bends per edge at the cost of an increased area. On the other hand, not all 4-planar graphs admit orthogonal drawings with one bend per edge; however, testing whether a 4-planar graph admits such a drawing can be done in polynomial time [3]. In the context of metro-map visualization, several approaches have been proposed to produce metro-maps using octilinear or nearly-octilinear polylines; see e.g., [9,18,19].

Preliminaries. In our algorithms, we incrementally construct the drawings similar to the method of Kant [11]. We first employ the canonical order to cope with triconnected graphs. Then, we extend them to biconnected graphs using the SPQR-tree and to simply connected graphs using the BC-tree. In this section we briefly recall them.

Definition 1 (Canonical order [11]). For a given triconnected plane graph $G = (V, E)$ let $\Pi = (P_0, \dots, P_m)$ be a partition of V into paths such that $P_0 = \{v_1, v_2\}$, $P_m = \{v_n\}$ and $v_2 \rightarrow v_1 \rightarrow v_n$ is a path on the outer face of G . For $k = 0, \dots, m$ let G_k be the subgraph induced by $\cup_{i=0}^k P_i$ and assume it inherits its embedding from G . Partition Π is a canonical order of G if for each $k = 1, \dots, m-1$ the following hold: (i) G_k is biconnected, (ii) all neighbors of P_k in G_{k-1} are on the outer face, of G_{k-1} (iii) all vertices of P_k have at least one neighbor in P_j for some $j > k$. P_k is called a singleton if $|P_k| = 1$ and a chain otherwise.

Definition 2 (BC-tree). The BC-tree $\mathcal{B}$ of a connected graph G has a B-node for each biconnected component of G and a C-node for each cutvertex of G . Each B-node is connected with the C-nodes that are part of its biconnected component.

An SPQR-tree [8,5] provides information about the decomposition of a biconnected graph into its triconnected components. Every triconnected component is associated with a node μ in the SPQR-tree $\mathcal{T}$. The triconnected component itself is referred to as the *skeleton* of μ , denoted by $G_\mu^{skel} = (V_\mu^{skel}, E_\mu^{skel})$. We refer to the degree of a vertex $v \in V_\mu^{skel}$ in G_μ^{skel} as $\deg_\mu^{skel}(v)$. We say that μ is an *R-node*, if G_μ^{skel} is a simple triconnected graph. A bundle of at least three parallel edges classifies μ as a *P-node*, while a simple cycle of length at least three classifies μ as an *S-node*. By construction R-nodes are the only nodes of the same type that are allowed to be adjacent in $\mathcal{T}$. The leaves of $\mathcal{T}$ are formed by the *Q-nodes*. Their skeleton consists of two parallel edges; one of them corresponds to an edge of G and is referred to as *real edge*. The skeleton edges that are not real are referred to as *virtual edges*. A virtual edge e in G_μ^{skel} corresponds to a tree node μ' that is adjacent to μ in $\mathcal{T}$, more exactly, to another virtual edge e' in $G_{\mu'}^{skel}$. We assume that $\mathcal{T}$ is rooted at a Q-node. Hence, every skeleton (except the one of the root) contains exactly one virtual edge $e = (s, t)$ that has a counterpart in the skeleton of the parent node. We call this edge the *reference edge* of μ denoted by $ref(\mu)$. Its endpoints, s and t , are named the *poles* of μ denoted by $\mathcal{P}_\mu = \{s, t\}$. Every subtree rooted at a node μ of $\mathcal{T}$ induces a subgraph of G called the *pertinent graph* of μ that we denote by $G_\mu^{pert} = (V_\mu^{pert}, E_\mu^{pert})$. We abbreviate the degree of a node v in G_μ^{pert} with $\deg_\mu^{pert}(v)$. The pertinent graph is the subgraph of G for which the subtree describes the decomposition. The following lemmata provide useful properties of SPQR-trees. Due to lack of space, their proofs are given in [1].

Lemma 1. Let μ be a tree node that is not the root in the SPQR-tree $\mathcal{T}$ of a simple, biconnected, k -planar graph G and μ' its parent in $\mathcal{T}$. For $v \in \mathcal{P}_\mu$, it holds that $\deg_\mu^{pert}(v) \leq k-2$, if μ' is a P- or an R-node and $\deg_{\mu'}^{pert}(v) \leq k-1$ otherwise, i.e. μ' is an S- or a Q-node.

Lemma 2. In the SPQR-tree $\mathcal{T}$ of a planar biconnected graph $G = (V, E)$ with $\deg(v) \geq 3$ for every $v \in V$, there exists at least one Q-node that is adjacent to a P- or an R-node.

2 Octilinear Drawings of 4-Planar Graphs

In this section, we focus on octilinear drawings of 4-planar graphs. We first consider the triconnected case and then we extend it to biconnected and simply connected graphs.

The Triconnected Case. Let $G = (V, E)$ be a triconnected 4-planar graph and $\Pi = \{P_0, \dots, P_m\}$ be a canonical order of G . We momentarily neglect the edge (v_1, v_2) of the first partition P_0 of Π and we start by placing the second partition, say a chain $P_1 = \{v_3, \dots, v_{|P_1|+2}\}$, on a horizontal line from left to right. Since v_3 and $v_{|P_1|+2}$ are adjacent to v_1 and v_2 , we place v_1 to the left of v_3 and v_2 to the right of $v_{|P_1|+2}$. So, they form a single chain where all edges are drawn using horizontal line-segments that are attached to the east and west port at their endpoints. The case where P_1 is a singleton is analogous. Having laid out the base of our drawing, we now place in an incremental manner the remaining partitions. Assume that we have already constructed a drawing for G_{k-1} and we now have to place P_k , for some $k = 2, \dots, m-1$.

In case where $P_k = \{v_i, \dots, v_j\}$ is a chain of $j - i + 1$ vertices, we draw them from left to right along a horizontal line one unit above G_{k-1} . Since v_i and v_j are the only vertices that are adjacent to vertices in G_{k-1} , both only to one, we place the chain between those two as in Fig. 1a. The port used at the endpoints of P_k in G_{k-1} depends on the following rule: Let v'_i (v'_j , resp.) be the neighbor of v_i (v_j , resp.) in G_{k-1} . If the edge (v_i, v'_i) ((v_j, v'_j) , resp.) is the last to be attached to vertex v'_i (v'_j , resp.), i.e., there is no vertex v in $P_l \in \Pi$, $l > k$ such that $(v'_i, v) \in E$ ($(v'_j, v) \in E$, resp.), then we use the northern port of v'_i (v'_j , resp.). Otherwise, we choose the north-east port for (v_i, v'_i) or the north-west port for (v_j, v'_j) .

In case of a singleton $P_k = \{v_i\}$, we can apply the previous rule if the singleton is of degree three. However, if v_i is of degree four, then we may have to deal with an additional third edge (v_i, v) that connects v_i with G_{k-1} . However, we may assume that v lies between the other two endpoints, thus, we place v_i such that $x(v_i) = x(v)$. This enables us to draw (v_i, v) as a vertical line-segment; see Fig. 1b.

The above procedure is able to handle all chains and singletons except the last partition P_m , as v_n may have 4 edges pointing downwards. We exclude (v_n, v_1) and draw v_n as an ordinary singleton. Then, we shift v_1 to the left and up as in Fig. 1c and draw (v_1, v_n) as a horizontal-vertical segment combination. Vertex v_2 is analogously moved. The drawings of the remaining edges incident to v_n are depicted in Fig. 1c.

A *cut* is a y -monotone continuous curve that crosses only horizontal segments and divides the current drawing into a left and a right part. Since every edge, except the ones drawn as vertical line-segments, contains exactly one horizontal segment, we can shift the right part of the drawing that is defined by the cut further to the right while keeping the left part of the drawing on place and the result remains a valid octilinear drawing.

To compute the x-coordinates, we first assign consecutive x-coordinates to the first two partitions. We may have to stretch the drawing in two cases: (i) when we introduce a chain, say P_k , as it may not fit into the gap defined by its two adjacent vertices in G_{k-1} , and, (ii) when an edge that contains a diagonal segment is to be drawn, to prevent it from intersecting any horizontal-vertical combinations in the face below it. We can cope with both cases by horizontally stretching the drawing by a factor that is bounded by the current height of the drawing. Since the height of the resulting drawing is bounded by $|\Pi| = O(n)$, it follows that in the worst case its width is $O(n^2)$. Note that our algorithm produces drawings that have a linear number of bends in total (in particular, exactly $2|\Pi| = O(n)$ bends). One can prove that this bound is asymptotically tight (see [1]). We are now ready to state the main theorem of this subsection.

Fig. 1. (a) Horizontal placement of a chain $P_k = \{v_i, \dots, v_j\}$. (b) Placement of a singleton $P_k = \{v_i\}$ with degree four. (c) Final layout after repositioning v_1 and v_2 (the shape of the dotted edges can be obtained by extending the stubs until they intersect).

Theorem 1. *Given a triconnected 4-planar graph G , we can compute in $O(n)$ time an octilinear drawing of G with at most 1 bend per edge on an $O(n^2) \times O(n)$ integer grid.*

Proof. In order to keep the time complexity of our algorithm linear, we employ a simple trick. We assume that any two adjacent points of the underlying integer grid are by n units apart in the horizontal direction and by one unit in the vertical direction. This a priori ensures that all edges that contain a diagonal segment will not be involved in crossings and simultaneously does not affect the total area of the drawing, which asymptotically remains cubic. On the other hand, the advantage of this approach is that we can use the shifting method of Kant [11] to cope with the introduction of chains in the drawing, that needs $O(n)$ time in total by keeping relative coordinates that can be efficiently updated and computing the absolute values only at the last step. $\square$

The Biconnected Case. Consider a node μ in the rooted SPQR-tree $\mathcal{T}$ of G with poles $\mathcal{P}_\mu = \{s, t\}$. In the drawing of G_μ^{pert} , s should be located at the upper-left and t at the lower-right corner of the drawing's bounding box with a port assignment as in Fig.2a. We also assume that the edges incident to s (t , resp.) use the western (eastern, resp.) port at their other endpoint, except of the northern (southern, resp.) most edge which may use the north (south, resp.) port instead. In that case we refer to s and t as *fixed*; see $\bar{e}_s, \bar{e}_t$ in Fig.2a. We maintain the following invariants:

- IP-1: The width (height) of the drawing of μ is quadratic (linear) in the size of G_μ^{pert} . s is located at the upper-left; t at the lower-right corner of the drawing's bounding box.
- IP-2: If $\deg_\mu^{\text{pert}}(s) \geq 2$, s is fixed; t is fixed if $\deg_\mu^{\text{pert}}(t) = 3$ and μ 's parent is not the root.
- IP-3: The edges that are incident at s and t in G_μ^{pert} use the south, south-east and east ports at s and the north, north-west and west port at t , resp. If s or t is not fixed, incident edges are attached at their other endpoints via the west and east port, respectively. If s or t is fixed, the northern-most edge at s and the southern-most edge at t may use the north (south, resp.) port at its other endpoint.

The port assignment, i.e. IP-3, guarantees the ability to stretch the drawing horizontally even in the case where both poles are fixed. Furthermore, IP-2 is *interchangeable* in the following sense: If $\deg_\mu^{\text{pert}}(s) = 2$ and $\deg_\mu^{\text{pert}}(t) = 1$, then s is fixed but t is

Fig. 2. (a) Schematic view of the layout requirements. (b) Creating a nose at t . (c) First P-node subcase without an (s, t) -edge but s might be fixed in a child μ_1 . (d) Second P-node subcase with an (s, t) -edge where t might get fixed in a child μ_2 .

not. But, if we relabel s and t such that $t' = s$ and $s' = t$, then $\deg_{\mu}^{\text{pert}}(s') = 1$ and $\deg_{\mu}^{\text{pert}}(t') = 2$. By IP-2, we can create a drawing where both s' and t' are not fixed and located in the upper-left and lower-right corner of the drawing's bounding box. Afterwards, we mirror the resulting layout vertically and horizontally to obtain one where s and t are in their respective corners and not fixed. For a non-fixed vertex, we introduce an operation referred to as forming or creating a *nose*; see Fig.2b, where t has been moved downwards at the cost of a bend. As a result, its west port is no longer occupied.

First consider the case where μ is a P-node. If there is no (s, t) -edge, then we draw the children of μ from top to bottom such that a possible child in which s is fixed, is drawn topmost (see μ_1 in Fig.2c). If there is an (s, t) -edge, then we draw it at the top and afterwards the remaining children of μ (see Fig.2d). This is possible only if s is not fixed in any of the other children. Let μ' be such a potential child where s is fixed, i.e., $\deg_{\mu'}^{\text{pert}}(s) = 2$, and thus, the only child that remains to be drawn. Here, we use the property of interchangeability to “unfix” s in μ' . As a result s can form a nose, whereas t may now be fixed in μ' when $\deg_{\mu'}^{\text{pert}}(t) = 2$ holds, as in Fig.2d. However, then $\deg_{\mu}^{\text{pert}}(t) = 3$ follows. Notice that the presence of an (s, t) -edge implies that the parent of μ is not the root of $\mathcal{T}$, since this would induce a pair of parallel edges. Hence, by IP-2 we are allowed to fix t in μ . Port assignment and area requirements comply in both cases with our invariant properties.

In the case where μ is an S-node, we place the drawings of its children, say $\mu_1, \dots, \mu_{\ell}$ in a “diagonal manner” such that their corners touch as in Fig.3a. In case of Q-nodes being involved, we draw their edges as horizontal segments (see, e.g., (v_3, v_4) in Fig.3a). s and t inherit their port assignment and pertinent degree from μ_1 and μ_{ℓ} , respectively. So, s (t , resp.) is fixed in μ , if it is fixed in μ_1 (μ_{ℓ} , resp.). By IP-2, t is not allowed to be fixed in the case where the parent of μ is the root of $\mathcal{T}$. However, from Lemma 2 we can choose the root such that t is not fixed in that case, and thus, complies with IP-2. Since we only concatenated the drawings of the children, IP-1 and IP-3 are satisfied.

For the case where μ is an R-node with poles $\mathcal{P}_{\mu} = \{s, t\}$, we follow the idea of the triconnected algorithm and describe only the required modifications. We assume the worst case where no child of μ is a Q-node. Let μ_{uv} be the child that is represented by the virtual edge $(u, v) \in E_{\mu}^{\text{skel}}$. Due to Lemma 1, $\deg_{\mu_{uv}}^{\text{pert}}(u) \leq 2$ and $\deg_{\mu_{uv}}^{\text{pert}}(v) \leq 2$ holds. By IP-2 we may assume that either u or v is fixed in μ_{uv} and choose the first partition in the canonical ordering to be $P_0 = \{s, t\}$.

In case of a chain, say $P_k = \{v_i, \dots, v_j\}$ with two neighbors v'_i and v'_j in G_{k-1} , we have to replace two types of edges with the drawings of the corresponding children:

Fig. 3. (a) S-node with children $\mu_1, \dots, \mu_4$; μ_3 is a Q-node representing the edge (v_3, v_4) . Optional edges are drawn dotted. (b) Example for a chain $v_i, \dots, v_j$ with virtual edges representing $\mu_i, \dots, \mu_{j-1}$ in the R-node case. (c) Singleton v_i with possibly three incident virtual edges representing μ'_i, μ'_v, μ'_j . (d) Placing v_n and moving up s which might be fixed in μ_{sn} .

the edges $(v_i, v_{i+1}), \dots, (v_{j-1}, v_j)$ representing the children $\mu_i, \dots, \mu_{j-1}$ and (v'_i, v_i) ((v_j, v'_j) resp.) representing μ'_i (μ'_j resp.). We place the vertices of P_k on a horizontal line high enough above G_{k-1} so that every drawing fits in-between. Then, we insert the drawings aligned below the horizontal line and choose for $i \leq l < j$, v_l to be the fixed node in μ_l , whereas in μ'_i (μ'_j resp.), we set v_i (v_j resp.) to be fixed. So, for $i \leq l < j$, v_{l+1} may form a nose in μ_l pointing upwards while v'_i and v'_j form each one downwards as in Fig.3b. For the extra height and width, we stretch the drawing horizontally.

In case where $P_k = \{v_i\}$, $i \neq n$ is a singleton, we only outline the difference which is a possible third edge (v_i, v) to G_{k-1} representing say μ'_v . While the other two involved children, say μ'_i and μ'_j , are handled as in the chain-case, μ'_v requires extra height. We place v_i so that μ'_v fits below μ'_j as in Fig.3c. Notice that $\deg^{\text{pert}}_{\mu'_v}(v_i) = 1$ and by IP-2 both v_i and v are not fixed in μ'_v . So, forming a nose at v_i and v is feasible.

For the last singleton $P_k = \{v_n\}$, observe that since $s, t \in P_0$, both have not been fixed. As in the triconnected algorithm we move $s = v_1$ above v_n as in Fig.3d to accommodate the drawing of the child μ_{sn} represented by the edge (s, v_n) . Since we may require v_n to form a nose in μ_{sn} as in Fig.3d, we choose s to be fixed in μ_{sn} . By IP-2 we are allowed to fix s since t remains unfixed. Although some diagonal segments may force us to stretch the whole drawing by its height, the height of the drawing has been kept linear in the size of G^{pert}_{μ} . Since we increase the width by the height a constant number of times per step, the resulting width remains quadratic.

If there is a vertex $v \in V$ with $\deg(v) \leq 3$, then we root $\mathcal{T}$ at a Q-node μ that represents one of its three incident edges and orient the poles $\{s, t\}$ such that $t = v$. So, for the child μ' of μ follows $\deg^{\text{pert}}_{\mu'}(t) \leq 2$. If $\deg(v) = 4$ for every $v \in V$, then we root $\mathcal{T}$ at a Q-node that is not adjacent to an S-node, which exists due to Lemma 2. In both cases, we may form a nose with t pointing downwards.

Theorem 2. *Given a biconnected 4-planar graph G , we can compute in $O(n)$ time an octilinear drawing of G with at most 1 bend per edge on an $O(n^2) \times O(n)$ integer grid.*

Proof. The SPQR-tree $\mathcal{T}$ can be computed in $O(n)$ -time and its size is linear to the size of G [8]. The pertinent degrees of the poles at every node can be pre-computed by a bottom-up traversal of $\mathcal{T}$. Drawing a P-node requires constant time; S- and R-nodes require time linear to the size of the skeleton. However, the sum over all skeleton edges is linear, as every virtual edge corresponds to a tree node. $\square$

The Simply Connected Case. The main idea of our algorithm is to root the BC-tree at some arbitrary B-node. With exception of the root, every B-node contains a designated cut vertex that links it to the parent. Similar to the biconnected case, we define an invariant for the drawing of a subtree. The cut vertex that links the subtree to the parent is located in the upper left corner of the bounding box. Due to lack of space, we only state the main result; its proof is given in [1].

Theorem 3. *Given a connected 4-planar graph G , we can compute in $O(n)$ time an octilinear drawing of G with at most 1 bend per edge on an $O(n^2) \times O(n)$ integer grid.*

3 Octilinear Drawings of 5-Planar Graphs

In this section, we focus on planar octilinear drawings of 5-planar graphs. As in Section 2, we first consider the case of triconnected 5-planar graphs and then we extend our approach first to biconnected and then to the simply connected graphs.

The Triconnected Case. Let $G = (V, E)$ be a triconnected 5-planar graph and $\Pi = \{P_0, \dots, P_m\}$ be a canonical order of G . We place partitions P_0 and P_1 similar to the 4-planar case. Assume that we have already constructed a drawing for G_{k-1} which is stretchable in the following sense: If $e \in E(G_{k-1})$ is an edge incident to the outer face, then there is a cut which crosses e and can be utilized to horizontally stretch the drawing of G_{k-1} . In other words, one can define a cut through every edge incident to the outer face of G_{k-1} (*stretchability-invariant*).

If $P_k = \{v_i, \dots, v_j\}$ is a chain, it is placed exactly as in the case of 4-planar graphs, but with different port assignment. Among the northern available ports of vertex v'_i (v'_j , resp.), edge (v_i, v'_i) ((v_j, v'_j) , resp.) uses the eastern-most unoccupied port of v'_i (western-most unoccupied port of v'_j , resp.); see Fig.4a. If P_k does not fit into the gap between v'_i and v'_j in G_{k-1} , then we horizontally stretch G_{k-1} between v'_i and v'_j to ensure that the horizontal distance between v'_i and v'_j is at least $|P_k| + 1$. This can be done due to the stretchability-invariant, as both v'_i and v'_j are on the outer face of G_{k-1} . Potential crossings introduced by edges of P_k containing diagonal segments can be eliminated by employing similar cuts to the ones presented in the 4-planar case. So, we may assume that G_k is plane. Also, G_k complies with the stretchability-invariant, as one can define a cut that crosses any of the newly inserted edges of P_k and then follows one of the cuts of G_{k-1} that crosses an edge between v'_i and v'_j .

Fig. 4. (a) Horizontal placement of a chain $P_k = \{v_i, \dots, v_j\}$. (b) Placement of a singleton $P_k = \{v_i\}$ of degree five. (c) Final layout (the shape of the dotted edges can be obtained by extending the stubs until they intersect).

In case of a singleton $P_k = \{v_i\}$ of degree 5, we have to deal with two additional edges (called *nested*) that connect v_i with G_{k-1} , say (v_i, v) and (v_i, v') ; see Fig.4b. Such a pair of edges does not always allow vertex v_i to be placed along the next available horizontal grid line. A careful case analysis on the type of ports that are unoccupied at v and v' in conjunction with the fact that G_{k-1} is horizontally stretchable shows that we can find a feasible placement for v_i . Potential crossings due to the remaining edges incident to v_i are eliminated by employing similar cuts to the ones presented in the 4-planar case. So, G_k is planar. Also, G_k complies with the stretchability-invariant. The last partition $P_m = \{v_n\}$ is treated in the same way, even if v_n can be incident to three nested edges. Since v_1 and v_2 are along a common horizontal line, (v_1, v_2) can be drawn using two diagonal segments that form a bend pointing downwards; see Fig.4c. Note that our algorithm may result in drawings of super-polynomial area, as proven in [1].

Theorem 4. *Given a triconnected 5-planar graph G , we can compute in $O(n^2)$ time an octilinear drawing of G with at most one bend per edge.*

Proof. We can no longer use the shifting method of Kant [11], since the x - and y -coordinates are not independent. However, the computation of each cut can be done in linear time, which implies that our drawing algorithm needs $O(n^2)$ time in total. $\square$

The Biconnected Case. For the 4-planar case we defined several invariants in order to keep the area of the resulting drawings polynomial. Since we drop this requirement now we can define a simpler new invariant for the biconnected 5-planar case. When considering a node μ in $\mathcal{T}$ and its poles $\mathcal{P}_\mu = \{s, t\}$, then in the drawing of G_μ^{pert} , s and t are horizontally aligned at the bottom of the drawing's bounding box as in Fig.5a. If an (s, t) -edge is present, it can be drawn at the bottom. An (s, t) -edge only occurs in the pertinent graph of a P-node (and Q-node). We use the term *fixed* for a pole-node that is not allowed to form a nose. We maintain the following properties through the recursive construction process: In S- and R- nodes, s and t are not fixed. In P- and Q-nodes, only one of them is fixed, say s . But as in the 4-planar case, we may swap their roles.

If μ is a P-node, then it has at most 4 children; one of them might be a Q-node, i.e., an (s, t) -edge, which can be drawn at the bottom as a horizontal segment. Since P-nodes are not adjacent to each other in $\mathcal{T}$, the remaining children are S- or R-nodes. By our invariant we may form noses enabling us to stack them as in Fig.5b.

Fig. 5. (a) Layout specification; s and t are located at the bottom. (b) P-node with an (s, t) -edge from a Q-node μ_1 . s and t form a nose in μ_2, μ_3 . (c) S-node example with four children $\mu_1, \dots, \mu_4$.

In case where μ is an S-node, we align its children $\mu_1, \dots, \mu_l$ horizontally; see Fig.5c. The poles inherit their pertinent degree from the children. The same holds for the property of being fixed. However, by our new invariant this is forbidden, as it requires that s and t are not fixed. It is easy to see that when μ_1 is a P-node, s is fixed by the invariant in μ_1 . In this case, we swap the roles of the poles in μ_1 such that s is not fixed. However, the other pole of μ_1 , say v_1 , is fixed now. Since the skeleton of an S-node is a cycle of length at least three, $v_1 \neq t$. So, s and t are not fixed in the resulting drawing.

To compute a layout of an R-node μ , we employ the triconnected algorithm (with $s = v_1$ and $t = v_2$). Let μ_e be a child of μ that corresponds to virtual edge $e = (u, v)$ in G_μ^{skel} . Then, $\deg_{\mu_e}^{pert}(u), \deg_{\mu_e}^{pert}(v) \leq 3$. When inserting the drawing of $G_{\mu_e}^{pert}$, we require at most three consecutive ports at u and v for the additional edges. As the triconnected algorithm assigns ports in a consecutive manner based on the relative position of the endpoints, we modify the port assignment so that an edge may have more than one port assigned. To do so, we assign each edge $e = (u, v)$ in G_μ^{skel} a pair $(\deg_{\mu_e}^{pert}(u), \deg_{\mu_e}^{pert}(v)) \in \{1, 2, 3\}^2$ that reflects the number of ports required by this edge at its endpoints. Then, we extend the triconnected algorithm such that when a port of u is assigned to an edge $e = (u, v)$, $\deg_{\mu_e}^{pert}(u) - 1$ additional consecutive ports in clockwise or counterclockwise order are reserved. The direction depends on the different types of edges that we will discuss next.

The simplest type of edges are the ones among consecutive vertices v_i, v_{i+1} of a chain. For each such edge we reserve the additional ports at v_i in counterclockwise and at v_{i+1} in clockwise order; see Fig.6a. So, we can later plug the drawing of the children into the layout as in Fig.6b without forming noses. In the same manner, we reserve the ports for the edges that connect $P_k = \{v_i, \dots, v_j\}$ to v'_i and v'_j in G_{k-1} (where P_k is singleton or chain), i.e., at v_i clockwise, (v_j counter-clockwise, resp.) and at v'_i counter-clockwise (v'_j clockwise); see Fig.6c. In case where (v_i, v'_i) or (v_j, v'_j) is a virtual edge, we choose the poles such that v_i (v_j resp.) is fixed in $\mu_{(v_i, v'_i)}$ ($\mu_{(v_j, v'_j)}$ resp.). Thus, we can create a nose with v'_i (v'_j resp.). Having exactly the ports required at both endpoints, we insert the drawing by replacing the bend with a nose as in Fig.6d. The remaining edges from P_k to G_{k-1} in case of a singleton $P_k = \{v_i\}$ are handled similarly; see Fig.6. During the replacement of the edges, the fixed vertex is always the upper one. The only exception are the horizontal drawn edges of a chain, for which it does not matter which one is fixed. Finally, we root $\mathcal{T}$ at an arbitrarily chosen Q-node representing a real edge (s, t) . By our invariant we may construct a drawing with s and t at the bottom of the drawing's bounding box, so that (s, t) has a 90° bend downwards.

Fig. 6. (a) Virtual edge $e = (v_i, v_{i+1})$ connecting two consecutive vertices of a chain. At both endpoints the drawing of μ_e requires two ports. (b) Replacing e in (a) with the corresponding drawing of the child μ_e . (c) Example of an edge $e = (v_j, v'_j)$ that requires three ports at v_j and two at v'_j . (d) Inserting the drawing of μ_e into (c) with v_j being fixed and v'_j forming a nose. (e) Reserving ports for the nested edges. A single port for a real edge is reserved and then two ports for the virtual edge $e = (v_i, v)$. (f) Final layout after inserting the drawing of μ_e .

Theorem 5. *Given a biconnected 5-planar graph G , we can compute in $O(n^2)$ time an octilinear drawing of G with at most one bend per edge.*

Proof. The ability to rotate and scale suffices to extend the result from 4-planar to 5-planar at the expense of the area. Similar to the 4-planar case, computing $\mathcal{T}$ takes linear time. Hence, the overall runtime is governed by the triconnected algorithm. $\square$

The Simply Connected Case. Due to lack of space, we outline the differences in comparison to the 4-planar case in [1]. Here, we simply state the main theorem.

Theorem 6. *Given a connected 5-planar graph G , we can compute in $O(n^2)$ time an octilinear drawing of G with at most one bend per edge.*

4 A Note on Octilinear Drawings of 6-Planar Graphs

In this section, we show that it is not always possible to construct a planar octilinear drawing of a given 6-planar graph with at most one bend per edge.

Theorem 7. *There exists an infinite class of 6-planar graphs which do not admit planar octilinear drawings with at most one bend per edge.*

Sketch of Proof. Due to lack of space the detailed proof of this theorem is given in [1]. The main idea is to construct an infinite class of maximal 6-planar graphs, whose outer face is always delimited by exactly three vertices, say v, v' and v'' , such that $\deg(v) = \deg(v') = 6$ and $5 \leq \deg(v'') \leq 6$. Then, it is not difficult to prove that it is not feasible to draw all edges incident to the outer face with at most one bend per edge. $\square$

5 Conclusions

We presented algorithms for the construction of planar octilinear drawings with at most one bend per edge for 4- and 5-planar graphs. Our work raises several open problems: (i) Is it possible to construct planar octilinear drawings of 4-planar (5-planar) graphs with at most one bend per edge in $o(n^3)$ (polynomial, resp.) area? (ii) Does any triangle-free 6-planar graph admit a planar octilinear drawing with at most one bend per edge? (iii) What is the number of necessary slopes for bendless drawings of 4-planar graphs?

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