A Crossing Lemma for the Pair-Crossing Number

Eyal Ackerman[1] and Marcus Schaefer[2]

Dept. Math., Physics, and Comp. Sci., University of Haifa at Oranim, Tivon, Israel
ackerman@sci.haifa.ac.il
School of Computing, DePaul University, Chicago, Illinois 60604, USA
mschaefer@cdm.depaul.edu

Abstract. The *pair-crossing number* of a graph G, $\mathrm{pcr}(G)$, is the minimum possible number of pairs of edges that cross each other (possibly several times) in a drawing of G. It is known that there is a constant $c \geq 1/64$ such that for every (not too sparse) graph G with n vertices and m edges $\mathrm{pcr}(G) \geq c\frac{m^3}{n^2}$. This bound is tight, up to the constant c. Here we show that $c \geq 1/34.2$ if G is drawn without adjacent crossings.

1 Introduction

Throughout this paper we consider graphs with no loops or parallel edges. A *topological graph* is a graph drawn in the plane with its vertices as distinct points and its edges as Jordan arcs that connect the corresponding points and do not contain any other vertex as an interior point. Every pair of edges in a topological graph has a finite number of intersection points. If every pair of its edges intersect at most once, then a topological graph is called *simple*. The intersection point of two edges is either a vertex that is common to both edges, or a crossing point at which one edge passes from one side of the other edge to its other side.

A *crossing* in a topological graph consists of a pair of crossing edges and a point in which they cross. The *crossing number* of a graph G, $\mathrm{cr}(G)$, is the minimum possible number of crossings in a drawing of G as a topological graph in the plane. The *pair-crossing number* of a graph G, $\mathrm{pcr}(G)$, is the minimum possible number of *pairs* of crossing edges in a drawing of G as a topological graph in the plane. There has been some confusion between these two notions in the literature, probably due to the fact that in a drawing with the least number of crossings no pair of edges intersects more than once. Perhaps for the same reason there has also been some confusion as to whether adjacent crossings are allowed or counted.[1] For examples and history of this confusion and other variants of the crossing number, see the recent survey of Schaefer [14] and the paper titled "Which crossing number is it anyway?" by Pach and Tóth [12].

Considering adjacent crossings, Pach and Tóth [11] introduced the following notation:

[1] By adjacent crossings we mean crossings between edges that share a common vertex.

C. Duncan and A. Symvonis (Eds.): GD 2014, LNCS 8871, pp. 222–233, 2014.

Rule +: adjacent crossings are not allowed.
Rule -: adjacent crossings are allowed but not counted.
Rule 0: adjacent crossings are allowed and counted (this is the default rule).

Clearly, $\mathrm{pcr}_-(G) \le \mathrm{pcr}(G) \le \mathrm{pcr}_+(G) \le \mathrm{cr}_+(G) = \mathrm{cr}(G)$ for every graph G. On the other hand, it is known [7] that $\mathrm{cr}(G) = O(\mathrm{pcr}(G)^{3/2} \log^2 \mathrm{pcr}(G))$, and it follows from the results in [13] that $\mathrm{cr}(G) \le \binom{2\mathrm{pcr}_-(G)}{2}$. Perhaps the main related open problem is to determine whether there is a graph G for which $\mathrm{pcr}(G) < \mathrm{cr}(G)$.

The following lower bound on the crossing number was proved by Ajtai, Chvátal, Newborn, Szemerédi [4] and, independently, by Leighton [6].

Theorem 1 ([4,6]). *There is an absolute constant $c > 0$ such that for every graph G with n vertices and $m \ge 4n$ edges we have $cr(G) \ge c\frac{m^3}{n^2}$.*

This celebrated result is known as the *Crossing Lemma* and has numerous applications in combinatorial and computational geometry, number theory, and other fields of mathematics. The Crossing Lemma is tight, apart from the multiplicative constant c. This constant was originally small, and later was shown to be at least $1/64 \approx 0.0156$, by a very elegant probabilistic argument due to Chazelle, Sharir, and Welzl [3]. Pach and Tóth [10] proved that $0.0296 \approx 1/33.75 \le c \le 0.09$ (the lower bound applies for $m \ge 7.5n$). Their lower bound was later improved by Pach, Radoičić, Tardos, and Tóth [9] to $c \ge 1024/31827 \approx 1/31.1 \approx 0.0321$ (when $m \ge \frac{103}{16}n$). Recently, Ackerman [1] further improved the lower bound to $c \ge \frac{1}{29}$ (when $m \ge 6.95n$).

Pach et al. [9] pointed out that the original proofs of the Crossing Lemma generalize to the pair-crossing number, yielding $\mathrm{pcr}(G) \ge \frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2}$ when $|E(G)| \ge 4|V(G)|$. They also remarked that they were unable to extend their lower bound on the crossing number to the pair-crossing number. Our main result is the following.

Theorem 2. *For every graph G with n vertices and $m \ge 6.75n$ edges we have $pcr_+(G) \ge \frac{2^6}{3^7}\frac{m^3}{n^2} \ge \frac{1}{34.2}\frac{m^3}{n^2}$.*

All the above-mentioned improvements for the crossing number were obtained using the same approach, namely, by showing that a sparse graph has an edge that is involved in several crossings. Denote by $e_k(n)$ the maximum number of edges in a topological graph with $n > 2$ vertices in which every edge is involved in at most k crossings. Let $e_k^*(n)$ denote the same quantity for *simple* topological graphs. It follows from Euler's Polyhedral Formula that $e_0(n) \le 3n-6$. Pach and Tóth showed that $e_k^*(n) \le 4.108\sqrt{k}n$ and also gave the following better bounds for $k \le 4$.

Theorem 3 ([10]). $e_k^*(n) \le (k+3)(n-2)$ *for* $0 \le k \le 4$. *Moreover, these bounds are tight when* $0 \le k \le 2$ *for infinitely many values of* n.

Pach et al. [9] observed that the upper bound in Theorem 3 applies also for not necessarily simple topological graphs when $k \le 3$, and proved a better bound for $k = 3$, namely, $e_3(n) \leq 5.5n - 11$. Ackerman [1] proved that $e_4^*(n) \leq 6n - 12$. The last two bounds are tight up to an additive constant.

The bounds $e_k(n)$ are used to get a weak lower bound on the crossing number of the form $\mathrm{cr}(G) \geq \alpha|E(G)| - \beta|V(G)|$. This linear bound is then used instead of the trivial bound $\mathrm{cr}(G) \geq |E(G)| - 3|V(G)|$ in the well-known probabilistic proof of the Crossing Lemma. The same approach would work to get a better lower bound for the pair-crossing number (and its variant pcr_+), if one can show that a sparse graph has an edge that crosses several other edges (each of them possibly many times).

Denote by $e''_k(n)$ the maximum number of edges in a topological graph with $n > 2$ vertices in which every edge crosses at most k other edges (each of them possibly more than once).[2] Clearly, $e''_k(n) \geq e_k(n) \geq e^*_k(n)$. For $0 \leq k \leq 3$ we have the following upper bounds on $e''_k(n)$.

Theorem 4. *Let G be a graph with $n \geq 3$ vertices that can be drawn as a topological graph in which every edge crosses at most k other edges (each of them possibly more than once). If $0 \leq k \leq 3$, then G has at most $(k + 3)(n - 2)$ edges.*

Note that for $0 \leq k \leq 2$ it is known that there are infinitely many values of n for which one can draw a (simple) topological graph with n vertices and $(k + 3)(n - 2)$ edges such that every edge is crossed at most k times. Therefore, the bounds for $0 \leq k \leq 2$ in Theorem 4 are tight. On the other hand, our upper bound for $e''_3(n)$ is inferior to the known bound on $e_3(n)$.

Organization. In Section 2 we collect some useful facts towards proving Theorem 4. A sketch of the proof of this theorem is then presented in Section 3. In Section 4 we recall how such a result can be used to get a better bound for the pair-crossing number when adjacent crossings are not allowed. Due to space limitation, some of the proofs are omitted or only sketched. The missing details can be found in the full version of the paper.

2 Preliminaries

In this section we collect some useful facts towards proving Theorem 4. Since we will be interested in the number of crossing pairs and the number of edges crossing a single edge, we may assume henceforth that the topological graphs that we consider have no three edges crossing at a single point. Indeed, if more than two edges cross at a point p, then we can redraw these edges in a small neighborhood of p such that no three of them cross at a point without changing the set of edges that each of these edges cross.

Recall that Pach et al. [9] proved that $e^*_k(n) = e_k(n)$, for $0 \leq k \leq 3$. This is implied by the following lemma.

[2] Think of the double prime symbol as a *pair* of prime symbols. Note that adjacent crossings are allowed and counted although for improving the crossing lemma for pcr_+ it would suffice to consider drawings without adjacent crossings.

Lemma 1 ([9]). *For every $0 \leq k \leq 3$, if a graph can be drawn as a topological graph such that each of its edges is crossed at most k times, then in a drawing with that property and the least number of crossings every pair of edges intersects at most once.*

Let G be a graph and let D be a drawing of G as a topological graph. Let D' be a drawing of G as a topological graph with the least number of crossings such that every pair of crossing edges in D' are also crossing in D. The following is implied by the proof of Theorem 3.2 in [15].

Lemma 2. *There is no pair of edges e and e' in D' such that there are two crossing points between them that are consecutive along e.*

Lemma 3. *If every edge in D (and hence in D') is crossed by at most k edges, then every edge in D' contains fewer than 2^k crossing points with other edges.*

3 Proof of Theorem 4

Recall that we wish to show that $e''_k(n) \leq (k+3)(n-2)$, for $0 \leq k \leq 3$. That is, for every $0 \leq k \leq 3$, if a graph G with $n > 2$ vertices can be drawn such that each of its edges crosses at most k edges (possibly several times), then G has at most $(k+3)(n-2)$ edges. Theorem 3 (due to Pach and Tóth) yields this edge bound for $k \leq 3$, but under the stronger assumption that each edge is crossed at most k times. As we see in the next section, for $k \leq 2$, the assumption is not really stronger, so that we can use Theorem 3 for these cases (though for $k = 0$ and $k = 1$ direct proofs are easier). For $k = 3$ we need a more sophisticated discharging argument, detailed in Section 3.2.

3.1 The Local Pair-Crossing Number and Bounding e''_k for $k \leq 2$

The *local crossing number*, $\mathrm{lcr}(G)$, of a graph G is the smallest k so that G can be drawn with at most k crossings per edge. The *local pair-crossing number*, $\mathrm{lpcr}(G)$, is the smallest k so that G has a drawing in which each edge is crossed by at most k other edges. By definition, $\mathrm{lpcr}(G) \leq \mathrm{lcr}(G)$. Following the hints in [14], it is possible to construct a graph G for which $\mathrm{lpcr}(G) = 4$ and $\mathrm{lcr}(G) = 5$, therefore, the two local crossing numbers differ. This is in marked contrast to the pair crossing number, which we cannot at this point separate from the standard crossing number.

Theorem 5. *If* $\mathrm{lpcr}(G) \leq 2$, *then* $\mathrm{lpcr}(G) = \mathrm{lcr}(G)$.

This leaves open the question whether equality holds for $\mathrm{lpcr}(G) = 3$. We believe a counterexample is possible, implying that we cannot take the easy route via the local crossing number to establish the bound of e''_3.

Proof. The statement is immediate for $\mathrm{lpcr}(G) = 0$ (by definition), and follows from Lemma 2 for $\mathrm{lpcr}(G) \leq 1$. Therefore, suppose that G is a graph with

$\mathrm{lpcr}(G) = 2$. Fix a drawing D of G in which every edge crosses at most two other edges and (under this condition) the least possible number of crossings.

It follows from Lemma 3 that every edge in D is crossed at most three times. We claim that every edge in D is crossed at most twice. Suppose for the sake of contradiction that there is an edge e that is crossed exactly three times. Orient e arbitrarily and let x_1, x_2, x_3 be the crossing points on e, in the order they appear on e according to its orientation. Denote by e_1, e_2, e_3 the edges that cross e at x_1, x_2, x_3, respectively. It follows from Lemma 2 that $e_1 = e_3$ and $e_1 \neq e_2$. Moreover, by the same argument, the segment of e_1 between x_1 and x_3 must contain a crossing point of e_1 with an edge e'. Note also that e' crosses e_1 once and e crosses e_2 once, since e and e_1 cross each other twice and there are at most three crossing points on every edge. Denote by D' the topological graph we obtain by swapping the segments of e and e_1 between x_1 and x_3 and redrawing them at small neighborhoods of x_1 and x_3 such that these segments are disjoint (see Fig. 1 for an illustration). Note that in D' every edge still crosses at most

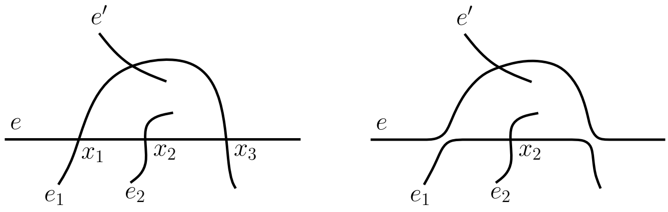


Fig. 1. Decreasing the number of crossings in case of an edge that is crossed three times

two other edges, however, D' has fewer crossing points than D, contradicting the minimality of D.

Thus, every edge in D is crossed at most twice, showing that $\mathrm{lcr}(G) \leq 2$. $\square$

For $k \leq 2$, we can now conclude that $e''_k(n) \leq (k+3)(n-2)$ as follows: By Theorem 5, if a graph G can be drawn so that each of its edges crosses at most $k \leq 2$ other edges, then G has a drawing in which each of its edges is crossed at most k times; by Lemma 1, we can assume that G has such a drawing which is simple. Now Theorem 4 yields the desired bound $e''_k(n) \leq e^*(n) = (k+3)(n-2)$.

The local pair-crossing number seems to be a useful tool for approaching arguments about the pair-crossing number, so we would like to know more about its properties. For example, one can ask whether lcr can be bounded in lpcr? This is true, by Lemma 3, which yields the exponential bound $\mathrm{lcr}(G) < 2^{\mathrm{lpcr}(G)}$, however, we think that much better bounds should be true, since we have much more flexibility with the local pair-crossing number than with string graphs.

In particular, it would bear investigating whether the upper bounds achieved by Tóth and Matoušek (bounding the crossing number in a function of the pair-crossing number), aren't really arguments about the local version of these crossing numbers.

3.2 A Proof that $e''_3(n) \le 6(n-2)$

Suppose that G is a graph with $n > 2$ vertices that can be drawn such that every edge crosses at most three other edges (possibly several times), and let D be a drawing of G as a topological graph with that property and the least possible number of crossings.

We prove that G has at most $6n-12$ edges by induction on n. For $n \le 10$ we have $6n-12 \le \binom{n}{2}$ and thus the theorem trivially holds. Therefore, we assume that $n \ge 11$. Furthermore, we may assume that the degree of every vertex in G is at least 7, for otherwise the theorem easily follows by removing a vertex of small degree and applying induction.

We denote by $M(D)$ the plane map induced by D. That is, the vertices of $M(D)$ are the vertices and crossing points in D, and the edges of $M(D)$ are the crossing-free segments of the edges of D (where each such edge-segment connects two vertices of $M(D)$). If a certain vertex of $M(D)$ is known to be a vertex of D we will denote it by a capital letter, otherwise, we will use small letters. Unless the context is clear, we will refer to edges of D as D-edges and to edges of $M(D)$ as *M-edges*. An edge-segment will be denoted by a concatenation of its endpoints (no comma in between), where round or square brackets will indicate whether an endpoint is included in the edge-segment. E.g., $[xy)$ is an edge-segment whose endpoints are x and y, such that x is included in $[xy)$ and y is not. In all the following figures bold edge-segments mark M-edges.

Proposition 1. *If $M(D)$ is not 2-connected, then D has at most $6n-12$ edges.*

By Proposition 1, we may assume henceforth that $M(D)$ is 2-connected. The *boundary* of a face f in $M(D)$ consists of all the M-edges that are incident to f. Since $M(D)$ is 2-connected, the boundary of every face in $M(D)$ is a simple cycle. Thus, we can define the *size* of a face f, $|f|$, as the number of M-edges on its boundary. It can be shown that if $M(D)$ contains a face of size two, then there is a drawing of G as a topological with fewer crossings than D and such that every edge crosses at most three other edges. Therefore, the size of every face in $M(D)$ is at least three.

We use the *Discharging Method* (see, e.g., [5]) to prove that $|E(D)| \le 6(n-2)$. We begin by assigning a *charge* to every face of the planar map $M(D)$ such that the total charge is $4n-8$. Then, we redistribute the charge in several steps such that eventually the charge of every face is nonnegative and the charge of every vertex $A \in V(D)$ is $\frac{1}{3}\deg(A)$. Hence, $\frac{2}{3}|E(D)| = \sum_{A \in V(D)} \frac{1}{3}\deg(A) \le 4n-8$ and we get the desired bound on $|E(D)|$. Next we describe the proof in details.

Charging. Let V', E', and F' denote the vertex, edge, and face sets of $M(D)$, respectively. For a face $f \in F'$ we denote by $V(f)$ the set of vertices of D that are incident to f. It is easy to see that $\sum_{f \in F'} |V(f)| = \sum_{A \in V(D)} \deg(A)$ and that $\sum_{f \in F'} |f| = 2|E'| = \sum_{u \in V'} \deg(u)$. Note also that every vertex in $V' \setminus V(D)$ is a crossing point in D and therefore its degree in $M(D)$ is four. Hence,

$$\sum_{f \in F'} |V(f)| = \sum_{A \in V(D)} \deg(A) = \sum_{u \in V'} \deg(u) - \sum_{u \in V' \setminus V(D)} \deg(u) = 2|E'| - 4\left(|V'| - n\right).$$

Assigning every face $f \in F'$ a charge of $|f|+|V(f)|-4$, we get that total charge over all faces is

$$\sum_{f \in F'} (|f| + |V(f)| - 4) = 2|E'| + 2|E'| - 4\,(|V'| - n) - 4|F'| = 4n - 8,$$

where the last equality follows from Euler's Polyhedral Formula by which $|V'| + |F'| - |E'| = 2$ (recall that $M(D)$ is connected).

Discharging. We will redistribute the charges in several steps. We denote by $ch_i(x)$ the charge of an element x (either a face in F' or a vertex in $V(D)$) after the ith step, where $ch_0(\cdot)$ represents the initial charge function. We will use the terms *triangles*, *quadrilaterals* and *pentagons* to refer to faces of size 3, 4 and 5, respectively. An integer before the name of a face denotes the number of original vertices it is incident to. For example, a 2-triangle is a face of size 3 that is incident to 2 original vertices.

Step 1: Charging the Vertices of D. In this step every vertex of D takes 1/3 units of charge from each face it is incident to. $\leftrightsquigarrow$

After Step 1 the charge of every vertex $A \in V(D)$ is $\frac{1}{3}\deg(A)$. Next, we need to make sure that the charge of every face is nonnegative. Let $f \in F'$ be a face. Note that $ch_1(f) \geq |f| + \frac{2}{3}|V(f)| - 4$ and therefore $ch_1(f) \geq 0$ if $|f| \geq 4$. Recall that $M(D)$ has no faces of size two. Thus, it remains to consider the case that f is a triangle: if f is a 3-triangle, then $ch_1(f) = 1$; if f is a 2-triangle, then $ch_1(f) = \frac{1}{3}$; if f is a 1-triangle, then $ch_1(f) = -\frac{1}{3}$; and if f is a 0-triangle, then $ch_1(f) = -1$.

In order to describe the way the charge of 0- and 1-triangles becomes nonnegative, we will need the following definitions. Let f be a face, let e be one of its edges, and let f' be the other face that shares e with f. We say that f' is the *immediate neighbor* of f at e.

Let f_0 be a face in $M(D)$ and let x_1 and y_1 be two vertices of f_0 that are consecutive on its boundary and are crossing points in D. Denote by e_1 (resp., e_2) the D-edge that crosses the D-edge that contains the edge-segment $[x_1y_1]$ at x (resp., y). Notice that it follows from Lemma 2 that e_1 and e_2 are distinct D-edges. Let f_1 be the immediate neighbor of f_0 at $[x_1y_1]$. For $i \geq 1$, if f_i is a 0-quadrilateral, then denote by $[x_{i+1}y_{i+1}]$ the edge opposite to $[x_iy_i]$ in f_i, such that e_1 contains x_{i+1} and e_2 contains y_{i+1}, and let f_{i+1} be immediate neighbor of f_i at $[x_{i+1}y_{i+1}]$ (see Fig. 2 for an illustration).

Clearly, it is impossible that $x_j = x_k$ or $y_j = y_k$ for $j \neq k$, since e_1 and e_2 do not cross themselves. Suppose that $x_j = y_k$ for some j and k. Assume without loss of generality that $j \leq k$. It cannot happen that $j = k$ for then f_{j-1} is not a 0-quadrilateral. Since $x_j = y_k$, e_2 crosses e_1 at x_j. The M-edges $[x_{j-1}x_j]$ and $[x_jx_{j+1}]$ are contained in e_1 (note that $[x_jx_{j+1}]$ exists since $j < k$). Therefore, e_2 contains $[x_jy_j]$. However, this implies that e_2 crosses itself at y_j (see Fig. 2 for an illustration).

It follows that it cannot happen that there are two 0-quadrilaterals f_i and f_j such that $i \neq j$ and $f_i = f_j$. By Lemma 3 every edge in D contains at most

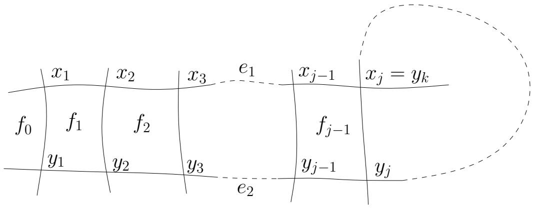


Fig. 2. If $x_j = y_k$, then e_2 crosses itself

seven crossing points. Therefore there must be an index $1 \leq k \leq 7$ such that f_k is not a 0-quadrilateral (notice that if f_i is not a 0-quadrilateral, then f_{i+1} is not defined). We say that f_k is the *distant neighbor* of f_0 at $[x_1y_1]$, and that the M-edge $[x_ky_k]$ is the edge of f_k that *faces* $[x_1y_1]$. Note that x_k and y_k must be crossing points, since they belong to the 0-quadrilateral f_{k-1} or coincide with x_1 and y_1, if $k = 1$. It is also important to note that if f_0 is not a 0-quadrilateral, then f_0 is the distant neighbor of f_k at $[x_ky_k]$ and $[x_1y_1]$ is the edge of f_0 that faces $[x_ky_k]$. Indeed, this follows from the definition of a distant neighbor and from the fact that the relation immediate neighbor at a certain M-edge and the relation opposite edge in a 0-quadrilateral are one-to-one.

Proposition 2. *Let t be a* 0- *or* 1*-triangle whose vertices are x, y and z, such that x and y are crossing points in D, and let e_1 (resp., e_2) be the D-edge that contains $[zx]$ (resp., $[zy]$). Suppose that f is the distant neighbor of t at $[xy]$ and e' is the edge of f that faces $[xy]$. Then:*

1. *the endpoints of e' are crossing points in D;*
2. *one endpoint of e' (denote it by p) lies on e_1 and the other endpoint of e' (denote it by q) lies on e_2;*
3. *t is the distant neighbor of f at $[pq]$, and $[xy]$ is the edge of t that faces $[pq]$;*
4. *the edge-segment $(zp]$ of e_1 (resp., $(zq]$ of e_2) does not intersect e_2 (resp., e_1); and*
5. *$|f| \geq 5$ or $|f| = 4$ and $|V(f)| \geq 1$.*

Step 2: Charging 0-triangles. Let t be a 0-triangle, let e be one of its edges, let f be the distant neighbor of t at e, and let e' be the edge of f that faces t. We move $1/3$ units of charge from f to t, and say that f *contributed* $1/3$ units of charge to t *through* e'.

In a similar way t obtains $1/3$ units of charge from each of its distant neighbors at its other edges. ⇜⇝

After the second discharging step the charge of every 0-triangle becomes zero. It remains to deal with 1-triangles, and then to make sure that the charge of every face did not become negative after the discharging steps. Let t be a 1-triangle and let $A \in V(D)$ be the vertex of D that is incident to t. Let g be an immediate neighbor of t that is incident to A. We call g a *good* neighbor of

t if $ch_2(g) > 0$, and a *bad* neighbor if $ch_2(g) \leq 0$. Note that t has two distinct (good/bad) neighbors, for otherwise $\deg(A) = 2 < 7 \leq \delta(G)$ or A is cut vertex in $M(D)$.

Step 3: Charging 1-triangles. Let t be a 1-triangle, let f be the distant neighbor of t and let e be the edge of f that faces t. (a) Every good neighbor of t contributes 1/6 units of charge to t through the M-edge that they share. (b) If after Step 3(a) the charge of t is still negative, then f contributes 1/6 units of charge to t through e. ⇜

See Fig. 3 for an illustrations of the discharging steps.

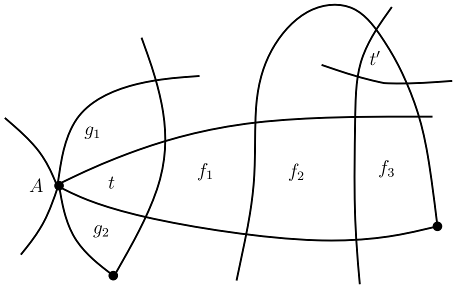


Fig. 3. Discharging steps: in Step 1 each of t, g_1 and g_2 contributes 1/3 units of charge to A; in Step 2 f_3 contributes 1/3 units of charge to t'; in Step 3(a) g_2 contributes 1/6 units of charge to t; and in Step 3(b) f_3 contributes 1/6 units of charge to t.

Considering Steps 2 and 3 and Proposition 2, we have the following observations.

Observation 3. *Let f be a face that contributes charge through one of its M-edges $[xy]$, and let z (resp., w) be the other vertex of f that is adjacent to x (resp., y). If x (resp., y) is a crossing point, then denote by (X, X') (resp., (Y, Y')) the D-edge that contains xz (resp., yw) such that $x \in [Xz]$ (resp., $y \in [Yw]$). We have:*

1. *f contributes charge through $[xy]$ exactly once;*
2. *if f contributes charge through $[xy]$ in Step 3(a), then one of x and y is a crossing point and the other is a vertex of D; and*
3. *if f contributes charge through $[xy]$ in Step 2 or Step 3(b), then both x and y are crossings points and f is a distant neighbor of a 0- or 1-triangle t. Furthermore, $[Xx]$ and $[Yy]$ intersect at a point q that is a vertex of t, (qx) and (qy) do not intersect, and $q = X = Y$ if t is a 1-triangle (otherwise, q is a crossing point).*

Recall that our plan was to distribute the initial charge such that the charge of every original vertex is one third of its degree and the charge of every face is nonnegative.

Lemma 4. *For every vertex $A \in V(D)$ we have $ch_3(A) = \frac{1}{3}\deg(A)$ and for every face $f \in F'$ we have $ch_3(f) \geq 0$.*

Proof. (sketch) The first part of the claim follows from the first discharging step. Let f be a face in $M(D)$. Since f contributes at most 1/3 units of charge through each of its edges, we have $ch_3(f) \geq \frac{2}{3}|f| + \frac{2}{3}|V(f)| - 4$. Therefore, if $|f| \geq 6$, then $ch_3(f) \geq 0$. Recall that there are no faces of size two in $M(D)$, thus, it remains to consider faces of size three, four and five, i.e., triangles, quadrilaterals and pentagons.

Triangles. Suppose that $|f| = 3$. It is easy to show that $ch_3(f) \geq 0$ if f is not a 1-triangle. If f is a 1-triangle, then we show that after Step 3(a) its charge is at least $-1/6$, since it cannot have two bad neighbors (for otherwise, there is an edge of D crossing four other edges). It then follows from Step 3(b) that the final charge of f is zero.

Quadrilaterals. Suppose that $|f| = 4$. A 0-quadrilateral does not contribute charge and therefore if $|V(f)| = 0$ we have $ch_3(f) = 0$. If $|V(f)| \geq 2$, then it is easy to see that $ch_3(f) \geq 0$. It remains to consider the case that f is a 1-quadrilateral. Observe that if f is a 1-quadrilateral and $ch_3(f) < 0$, it must be that f contribute 1/3 units of charge through precisely one of its edges in Step 2, and 1/6 units of charge through each of its other edges. However, this case implies that there is an edge of D that crosses four other edges (see Fig. 4 for an illustration).

Pentagons. Suppose that $|f| = 5$. Recall that $ch_3(f) \geq \frac{2}{3}|f| + \frac{2}{3}|V(f)| - 4$. Therefore, if $|V(f)| \geq 1$, then $ch_3(f) \geq 0$, and it remains to consider the case that f is a 0-pentagon. Recall that we may assume that the boundary of f consists of a simple 5-cycle. It follows from Lemma 2 that all the D-edges that contain the M-edges of f are distinct. From this fact and Observation 3 one concludes that it is impossible that f contributes charge through two of its edges that are not consecutive on its boundary. Thus, if $ch_3(f) < 0$, then f must contribute 1/3 units of charge through two (consecutive) edges in Step 2 and

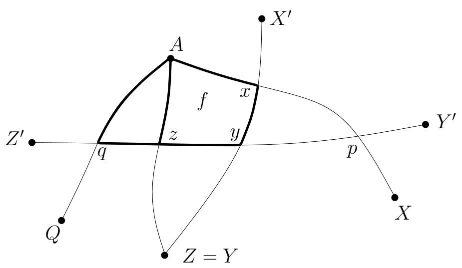
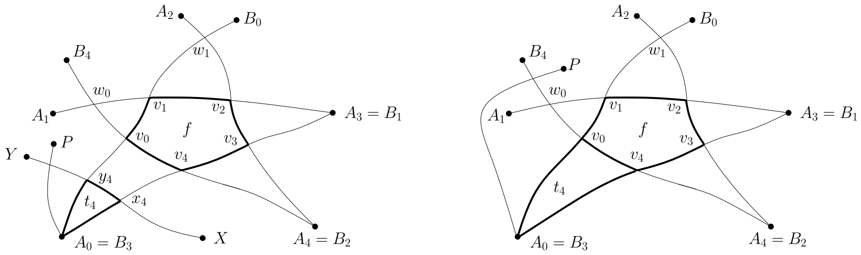


Fig. 4. If f is a 1-quadrilateral that contributes 1/3 units of charge through $[xy]$ and 1/6 units of charge through each of $[Az]$ and $[yz]$, then (Y', Z') crosses four edges



(a) If $[v_4 v_0]$ is not an edge of t_4, then there are four D-edges that cross (A_0, B_0).

(b) If $[v_4 v_0]$ is an edge of t_4, then there are four D-edges that cross (A_4, B_4).

Fig. 5. The 0-pentagon f contributes charge through $[v_0 v_1]$ and $[v_1 v_2]$ in Step 2 and through $[v_2 v_3]$, $[v_3 v_4]$ and $[v_4 v_0]$ in Step 3(b)

1/6 units of charge through each of its other three edges in Step 3(b). A simple case-analysis shows that this scenario is impossible, as it implies that there is an edge of D that crosses four other edges (see Fig. 5 for illustrations).

It follows from Lemma 4 that the final charge of every face in $M(D)$ is nonnegative and that the charge of every vertex of D equals to one third of its degree. Recall that the total charge is $4n-8$. Therefore, $\frac{2}{3}|E(D)| = \sum_{A \in V(D)} \frac{1}{3}\deg(A) \leq 4n-8$ and thus $|E(G)| = |E(D)| \leq 6n-12$. This concludes the proof of Theorem 4. □

4 A Better Lower Bound on $\mathbf{pcr}_+$

Recall that $e''_k(n)$ is the maximum number of edges in a topological graph with $n > 2$ vertices in which every edge crosses at most k other edges (each of them possibly more than once). Using the bounds on $e''_k(n)$ from Theorem 4, one can prove:

Theorem 6. *For every graph G with $n > 2$ vertices and m edges we have: $pcr(G) \geq m - 3(n-2)$; $pcr(G) \geq 2m - 7(n-2)$; $pcr(G) \geq 3m - 12(n-2)$; and $pcr(G) \geq 4m - 18(n-2)$.*

Using the new linear bounds it is now possible to obtain a better lower bound for pcr_+, following the probabilistic proof of the Crossing Lemma, as in [8,9,10].

Proof of Theorem 2: Let G be a graph with n vertices and $m \geq 6.75n$ edges and consider a drawing of G as a topological graph with $\text{pcr}_+(G)$ pairs of crossings edges and without adjacent crossings. Construct a random subgraph of G by selecting every vertex independently with probability $p = 6.75n/m \leq 1$. Let G' be the subgraph of G that is induced by the selected vertices. Denote by n' and m' the number of vertices and edges in G', respectively. Clearly, $\mathbb{E}[n'] = pn$ and $\mathbb{E}[m'] = p^2 m$. Denote by x' the number of pairs of crossing edges in the drawing of G' inherited from the drawing of G. Then $\mathbb{E}[\mathrm{pcr}_+(G')] \leq \mathbb{E}[x'] = p^4 \cdot \mathrm{pcr}_+(G)$.[3] It follows from Theorem 6 that $\mathrm{pcr}_+(G') \geq \mathrm{pcr}(G') \geq 4m' - 18n'$ (note that this it true for any $n' \geq 0$), and this holds also for the expected values: $\mathbb{E}[\mathrm{pcr}_+(G')] \geq 4\mathbb{E}[m'] - 18\mathbb{E}[n']$. Plugging in the expected values we get that $\mathrm{pcr}_+(G) \geq \frac{2^6}{3^7}\frac{m^3}{n^2} \geq \frac{1}{34.2}\frac{m^3}{n^2}$. $\square$

References

1. Ackerman, E.: On topological graphs with at most four crossings per edge (manuscript)
2. Ackerman, E., Tardos, G.: On the maximum number of edges in quasi-planar graphs. J. Combinatorial Theory, Ser. A 114(3), 563–571 (2007)
3. Aigner, M., Ziegler, G.: Proofs from the Book. Springer, Heidelberg (2004)
4. Ajtai, M., Chvátal, V., Newborn, M., Szemerédi, E.: Crossing-free subgraphs. In: Theory and Practice of Combinatorics. North-Holland Math. Stud., vol. 60, pp. 9–12. North-Holland, Amsterdam (1982)
5. Cranston, D.W., West, D.B.: A guide to discharging (manuscript)
6. Leighton, F.T.: Complexity Issues in VLSI: Optimal Layouts for the Shuffle-Exchange Graph and Other Networks. MIT Press, Cambridge (1983)
7. Matoušsek, J.: Near-optimal separators in string graphs, ArXiv (February 2013)
8. Montaron, B.: An improvement of the crossing number bound. J. Graph Theory 50(1), 43–54 (2005)
9. Pach, J., Radoičić, R., Tardos, G., Tóth, G.: Improving the crossing lemma by finding more crossings in sparse graphs. Disc. Compu. Geometry 36(4), 527–552 (2006)
10. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. Combinatorica 17(3), 427–439 (1997)
11. Pach, J., Tóth, G.: Thirteen problems on crossing numbers. Geombinatorics 9(4), 194–207 (2000)
12. Pach, J., Tóth, G.: Which crossing number is it anyway? J. Combin. Theory Ser. B 80(2), 225–246 (2000)
13. Pelsmajer, M.J., Schaefer, M., Štefankovič, D.: Removing independently even crossings. SIAM J. on Disc. Math. 24(2), 379–393 (2010)
14. Schaefer, M.: The graph crossing number and its variants: A survey. Electronic Journal of Combinatorics, Dynamic Survey 21 (2013)
15. Schaefer, M., Štefankovič, D.: Decidability of string graphs. J. Comput. System Sci. 68(2), 319–334 (2004)

[3] Here we need the restriction that there are no adjacent crossings: an adjacent crossing would survive with probability p^3 instead of p^4. This was overlooked in a preliminary version of this paper when this theorem was stated for pcr instead of pcr_+.