A GENERALIZED PROBABILISTIC ANALYSIS OF A POLYHEDRON'S NUMBER OF SHADOW VERTICES

Karl-Heinz Küfer, Universität Kaiserslautern

Introduction: Let $A := \{a_1, \dots, a_m\}$, $a_i \in \mathbb{R}^n$, $m \geq n \geq 2$, where vectors a_i are considered i.i.d.-vectors under an arbitrary rotationally symmetric distribution in $\mathbb{R}^n$. The random event A may be identified with a polyhedron X_A by $X_A := \{x \in \mathbb{R}^n \mid a^T x \leq 1, a \in A\}$. Furthermore, let L_k , $L_k := \text{span}(u_1, \dots, u_k)$, $u_i \in S^{n-1}$, be an arbitrary $\mathbb{R}^n$ -subspace of dimension $k \in \{1, \dots, n\}$. We call a vertex of X_A " L_k -shadow vertex" if it remains a vertex of X_A 's image under an orthogonal projection into L_k . The number of X_A 's L_k -shadow vertices will be denoted by $S_{L_k}(X_A)$. Knowledge about a polyhedron's number of shadow vertices is of interest within the probabilistic analysis of algorithms, where the cardinality of a polyhedron's vertex subset is the main benchmark of complexity. A famous example is the "shadow vertex algorithm" — a variant of the simplex algorithm — in case of planary projection spaces ($k=2$), cf. [1,2].

Results: We study expectation value and variance of the random variable $S_{L_k}(X_A)$ in two different ways, the first leading to asymptotic results for fixed dimension n , $m \rightarrow \infty$, and special distributions, the latter overestimating moments independently from the underlying distribution. In both cases we concentrate on the increasing rate in m and give only qualitative results about the growth in n , k is always assumed to be fixed.

In case of expectation values a typical asymptotic result among many others is the following:

$$(1) \quad E(S_{L_k}) = C_{n,k} m^{(k-1)/(n+1)} (1 + o(1)), \quad m \rightarrow \infty,$$

in case of a_i being uniformly distributed in the unit ball of $\mathbb{R}^n$, $n \geq 2$. Here, the constant $C_{n,k}$ is polynomial in n for fixed k . Independently from the underlying rotationally invariant distribution holds:

$$(2) \quad E(S_{L_k}) \leq C_{n,k} m^{(k-1)/(n-1)}, \quad m \geq n \geq 2,$$

$C_{n,k}$ being polynomial in n for fixed k again. Corresponding results for variances of S_{L_k} are

$$(3) \quad \frac{\text{Var}(S_{L_k})}{E^2(S_{L_k})} = \Theta(m^{-(k-1)/(n+1)}), \quad m \rightarrow \infty.$$

for uniformly distributed a_i in the n -dimensional unit ball and

$$(4) \quad \frac{\text{Var}(S_{L_k})}{E^2(S_{L_k})} \leq C_{n,k},$$

which holds independently from the underlying distribution. In (3) and (4) the constants $C_{n,k}$ are polynomial in n for fixed k . By the aid of Chebychev's inequality (3) implies that in case of large m and fixed n even small relative deviations from the mean are very rare. This observation is true for all rotationally invariant distributions with compact support, while there are distributions with non-compact support, where the investigated quotient in (3) and (4) is positively bounded from below.

References:

- [1] Borgwardt, K. H. (1982) *The Average Number of Pivot Steps Required by The Simplex Method is Polynomial*. ZOR 26: 157-177.
- [2] Küfer, K. H. (1992) *On the Variance of the Number of Pivot Steps Required by the Simplex Algorithm*. To appear in ZOR, Preprint Universität Kaiserslautern.