Parameter Estimation in Differential Equations, using Random Time Transformations

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Consider the differential equation $dX(t) = A(X(t))dt$, $X(t_0) = \text{given}$, where the function A depends on some unknown (vector) parameter θ to be estimated from a realization $\tilde{X}(t_0) = x_0, \tilde{X}(t_1) = x_1, \dots, \tilde{X}(t_n) = x_n$ of a process $\tilde{X}$ related to X but which incorporates measurement errors. As an example, consider the logistic equation for population growth in a finite environment and its solution

$$A(x) = rx(1 - x/k) ; \quad X(t) = k/(1 + (k/X(t_0) - 1)\exp\{-r(t - t_0)\}) .$$

Two of the most common approaches to the estimation of the unknown parameters in the presence of errors are Regression and Diffusion. We will follow an approach which models the randomness as pertaining inherently to the time lag between apparent time $X^{-1}(x_j)$ and chronological time t_j . The observed process $\tilde{X}(t)$ is viewed as $\tilde{X}(t) = X(T(t))$.

The rationale behind the random time transformation is illustrated by the following examples. The “age” of a machine is better modelled by the hours of operation than by the time elapsed since the date of purchase. If a plant disease is transmitted by insects, incidence of infection at time t within a closed field is better modelled by the total number of insect-hours than by t itself. If environmental or climatic local conditions increase temporarily the insect population, then “time” runs faster.

Brownian Motion time transformations correspond to a White Noise *rate of change* of time. A smoother Gaussian rate of change would lead to a non-Markovian time change, the Integrated Ornstein-Uhlenbeck time transformation.

We model multivariate random time transformations for the analysis of multiple-field data, consisting of

$$\tilde{X}_i(t_{i,j}) = X_i(T_i(t_{i,j})) = x_{i,j} ; \quad 1 \leq i \leq p ; \quad 0 \leq j \leq n_i$$

coming from p differential equations subject to a dependent noise structure. The Brownian Motion Time Transformation model $\mathbf{T}(t) = (T_1(t), T_2(t), \dots, T_p(t))$ assumes that the process $\mathbf{T}$ has stationary, independent increments, and that such an increment, say $(T_1(t + \Delta) - T_1(t), T_2(t + \Delta) - T_2(t), \dots, T_p(t + \Delta) - T_p(t))$, is jointly normally distributed with mean vector $(\Delta, \Delta, \dots, \Delta)$ and covariance matrix Σ . As we shall see, the maximization of the likelihood over Σ can be brought to the standard form of finding a (constrained or unconstrained) ML estimate of a covariance matrix based on an i.i.d. sample from a $N(0, \Sigma)$ distribution. Any existing package for such a purpose (such as LISREL) can serve as a “black box” yielding an objective function - the *profile likelihood* - in terms of the parameters θ exclusively.