Topological Operators on Cell Complexes in Arbitrary Dimensions

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Abstract. Cell complexes have extensively been used as a compact representation of both the geometry and topology of shapes. They have been the basis modeling tool for boundary representations of 3D shapes, and several dimension-specific data structures and modeling operators have been proposed in the literature. Here, we propose basic topological modeling operators for building and updating cell complexes in arbitrary dimensions. These operators either preserve the topology of the cell complex, or they modify it in a controlled way. We compare these operators with the existing ones proposed in the literature, in particular with handle operators and various Euler operators on 2D and 3D cell complexes.

Keywords: geometric modeling, cell complexes, topology-preserving operators, topology-modifying operators.

1 Introduction

Cell complexes, together with simplicial complexes, have been used as a modeling tool in a wide range of application domains, such as solid modeling, computer graphics, computer aided design, finite element analysis, animation, scientific visualization, and geographic data processing. They are used to discretize geometric shapes, such as static and dynamic 3D objects, or surfaces and hyper-surfaces describing the behavior of scalar or vector fields.

The literature on operators for building and updating cell complexes is vast but quite disorganized. A variety of topological operators have been designed for building and updating data structures representing 2D and 3D cell complexes, such as handle operators and Euler operators. Handle operators are based on handlebody theory, stating that any n -manifold can be obtained from an n -ball by attaching handles to it. The main characteristic of Euler operators is that they maintain the Euler-Poincaré formula expressing the topological validity condition of a cell complex.

We propose a set of Euler operators which form a minimally complete basis for building and updating cell complexes in arbitrary dimensions in a topologically consistent manner. We distinguish between operators that maintain the topology

of the complex, and the ones that modify it in a controlled manner. Topology-preserving operators add (or remove) a pair of cells of consecutive dimension, but they do not change the Betti numbers of the complex. Topology-modifying operators add (or remove) an i -cell, and increase (decrease) the i th Betti number, or decrease (increase) the $(i-1)$ st Betti number of Γ . We compare the 2D and 3D instance of our operators with other Euler operators, and with handle operators, proposed in the literature for 2D and 3D cell complexes, and we show how these latter can be expressed in terms of the Euler operators we define here.

In Section 2, we review some background notions on cell complexes. In Section 3, we introduce a set of topological operators for building and updating cell complexes in arbitrary dimensions. In Section 4 we show how Euler operators proposed in the literature can be expressed through 2D and 3D instances of our operators, and in Section 5, we show how handle operators can be expressed in terms of our operators. In Section 6, we draw some concluding remarks, and discuss possible research directions.

2 Background Notions

We review here some notions on cell complexes, that we will use throughout this paper (see [Ago05] for more details). A k -cell in the Euclidean space $\mathbb{E}^n$ is a homeomorphic image of a k -dimensional ball, and a *cell complex* in $\mathbb{E}^n$ is a finite set Γ of cells in $\mathbb{E}^n$ of dimension at most d , $0 \leq d \leq n$, such that

- the cells in Γ are pairwise disjoint,
- for each cell $\gamma \in \Gamma$, the boundary of γ is a disjoint union of cells of Γ .

If the maximum dimension of the cells in Γ is equal to d , then Γ is called a d -complex. The set of cells on the boundary of a cell γ is called the (*combinatorial*) *boundary* of γ . The (*combinatorial*) *co-boundary* (or *star*) of γ consists of all the cells of Γ that have γ on their combinatorial boundary. An h -cell γ' on the boundary of a k -cell γ , $0 \leq h \leq k$, is called an h -face of γ , and γ is called a *coface* of γ' . Each cell γ is a face of itself. If $\gamma' \neq \gamma$, then γ' is called a *proper face* of γ , and γ and γ' are said to be *incident*. The *domain*, or *carrier*, of a cell d -complex Γ embedded in E^n , with $0 \leq d \leq n$, is the subset of E^n defined by the union, as point sets, of all the cells in Γ .

The Euler-Poincaré formula expresses the necessary validity condition of a cell complex with manifold or non-manifold carrier [Ago05]. The Euler-Poincaré formula for a cell d -complex Γ (with or without boundary, of homogenous or non-homogenous dimension) with n_i i -cells states that

$$\sum_{i=0}^d (-1)^i n_i = n_0 - n_1 + \dots + (-1)^d n_d = \sum_{i=0}^d (-1)^i \beta_i = \beta_0 - \beta_1 + \dots + (-1)^d \beta_d.$$

Here, β_i is the i th Betti number of Γ , and it measures the number of independent non-bounding i -cycles in Γ , i.e., the number of independent i -holes. The alternating sum $n_0 - n_1 + \dots + (-1)^d n_d$ is denoted as $\chi(\Gamma)$, and is called the *Euler-Poincaré characteristic* of Γ .

3 Topological Operators

There have been many diverse proposals in the literature for manipulation operators on 2D and 3D cell complexes. We propose here a minimal set of Euler operators on cell complexes in arbitrary dimensions, which subsume all the other Euler operators proposed in the literature. We classify these operators as:

- topology-preserving operators: $MiC(i+1)C$ (*Make i-Cell and (i+1)-Cell*),
- topology-modifying operators: $MiCiCycle$ (*Make i-Cell and i-Cycle*).

There are in total d topology-preserving operators, and $(d+1)$ topology-modifying operators.

Topology-preserving operators $MiC(i+1)C$ change the number of cells in the complex Γ , by increasing the number n_i of i -cells and the number n_{i+1} of $(i+1)$ -cells by one. The inverse $KiC(i+1)C$ (*Kill i-Cell and (i+1)-Cell*) operators delete an i -cell and an $(i+1)$ -cell from Γ . The Euler characteristic and the Betti numbers of the complex remain unchanged. Topology-preserving operator $MiC(i+1)C$ can create two new cells from an existing i - or $(i+1)$ -cell, or insert the new cells in the complex. The first type of $MiC(i+1)C$ operator either splits an existing $(i+1)$ -cell into two by splitting its boundary, and creates an i -cell separating the two $(i+1)$ -cells, or dually, it splits an existing i -cell in two by splitting its co-boundary, and creates an $(i+1)$ -cell bounded by the two i -cells. The second type of $MiC(i+1)C$ operator either creates an i -cell and an $(i+1)$ -cell bounded only by the i -cell, or dually, it creates an $(i+1)$ -cell and an i -cell bounding only the $(i+1)$ -cell.

Topology-modifying operators change both the number of cells in the complex Γ and its Betti numbers, and thus they change the Euler characteristic of Γ . They increase the number n_i of i -cells and the number β_i of non-bounding i -cycles by one. The inverse $KiCiCycle$ (*Kill i-Cell and i-Cycle*) operators delete an i -cell and destroy an i -cycle, thus decreasing the numbers n_i and β_i by one.

We note here that the Betti numbers of a cell complex are determined by the immediate boundary relation ∂_i on the complex, which relates i -cells with $(i-1)$ -cells. The creation of an i -cell affects relations ∂_{i+1} and ∂_i , and thus it either increases β_i by one, or it decreases β_{i-1} by one, to maintain the validity of the Euler-Poincaré formula. In the first case, we obtain our operators $MiCiCycle$ (*Make i-Cell and i-Cycle*).

Operators $MiCK(i-1)Cycle$ (*Make i-Cell, Kill (i-1)-Cycle*), $i \geq 2$, obtained in the second case, can be expressed through the proposed ones as: $K(i-1)C(i-1)Cycle$ (*Kill (i-1)-Cell and (i-1)-Cycle*) applied on one $(i-1)$ -cell on the boundary of the i -cell to be created (possibly preceded by some $K(i-1)CiC$, which delete all i -cells in the co-boundary of the $(i-1)$ -cell), followed by $M(i-1)CiC$ (*Make (i-1)-Cell and i-Cell*), which re-creates the deleted $(i-1)$ -cell and creates the i -cell (followed by $M(i-1)CiC$, inverse to $K(i-1)CiC$, which restore the deleted cells).

$M1CK0Cycle$ (*Make 1-cell, Kill 0-Cycle*) can be obtained by destroying one of the merged 0-cycles (components), applying one $M0C1C$, and re-creating the destroyed component.

M0C0Cycle (Make 0-Cell and 0-Cycle, i.e., make vertex and connected component) is also an *initialization* operator, which creates a new complex Γ .

It can be shown that the proposed operators form a complete set of basis operators for creating and updating cell complexes, by interpreting these operators as ordered $(2d+2)$ -tuples $(x_0, x_1, \dots, x_d, c_0, c_1, \dots, c_d)$ in an integer grid, belonging to the hyperplane $\sum_{i=0}^d (-1)^i x_i = \sum_{i=0}^d (-1)^i c_i$ defined by the Euler-Poincaré formula. The first $d+1$ coordinates denote the number of i -cells created or deleted by the operator, depending on the sign of the coordinate, and the last $d+1$ coordinates denote the change in the Betti numbers of the complex induced by the operator. Operator $MiC(i+1)C$, $0 \leq i \leq d-1$, has coordinates $x_i = x_{i+1} = 1$, $x_j = 0$, $j \in \{0, 1, \dots, d\} \setminus \{i, i+1\}$, $c_j = 0$, $j \in \{0, 1, \dots, d\}$. Operator $MiCiCycle$, $0 \leq i \leq d$, has coordinates $x_i = c_i = 1$, $x_j = c_j = 0$, $j \in \{0, 1, \dots, d\} \setminus \{i\}$.

A linear combination $\sum_{i=0}^d \alpha_d MiC(i+1)C + \sum_{i=0}^d \beta_d MiCiCycle$ vanishes if and only if $\alpha_i = \beta_i = 0$, $0 \leq i \leq d$, implying that the tuples corresponding to our operators are linearly independent. Each tuple $(a_0, a_1, \dots, a_d, b_0, b_1, \dots, b_d)$ in the hyperplane can be expressed through the $2d+1$ independent $(2d+2)$ -tuples corresponding to our operators as $\sum_{i=0}^d \alpha_d MiC(i+1)C + \sum_{i=0}^d \beta_d MiCiCycle$, where

$\alpha_i = \sum_{j=0}^i (-1)^{i-j} a_j - \sum_{j=0}^i (-1)^{i-j} b_j$, and $\beta_i = b_i$, $0 \leq i \leq d$. Thus, each operator satisfying the Euler-Poincaré formula on a cell complex Γ can be expressed through our operators. In the space (hyperplane) of dimension $(2d+1)$, a generating set consisting of $(2d+1)$ independent tuples forms a basis for the hyperplane.

For a 2-complex Γ embedded in $\mathbb{E}^3$, the operators are also called:

- Topology-preserving operators: *MVE* (*Make Vertex and Edge*) and *MEF* (*Make Edge and Face*).
- Topology-modifying operators: *MV0Cycle* (*Make Vertex and 0-Cycle*), *ME1Cycle* (*Make Edge and 1-Cycle*) and *MF2Cycle* (*Make Face and 2-Cycle*).

Operator *MV0Cycle* creates a new vertex and a new connected component, it increases by one the number of vertices (0-cells) and the zeroth Betti number β_0 . It is also an initialization operator. Operator *ME1Cycle* creates a new edge and forms a 1-cycle, thus increasing by one the number of edges (1-cells) and the first Betti number β_1 . Operator *ME2Cycle* creates a new face and forms a 2-cycle, thus increasing by one the number of faces (2-cells) and the second Betti number β_2 . Figure 1 shows an example of *MV0Cycle*, *ME1Cycle* and *MF2Cycle* operators.

For a 3-complex Γ embedded in $\mathbb{E}^3$, there will be an additional topology-preserving operator *MFV1* (*Make Face and Volume (3-Cell)*) which adds a new face (2-cell) and a new three-dimensional (volumetric) cell. The topology-modifying operators will be the same as for 2-complexes, since in this case the third Betti number β_3 is null.

Fig. 1. Topology-modifying operators on a 2-complex in $\mathbb{E}^3$: *MV0Cycle* (Make Vertex and 0-Cycle) (a); *ME1Cycle* (Make Edge and 1-Cycle) (b); *MF2Cycle* (Make Face and 2-Cycle) (c)

4 Comparison with Other Euler Operators

We show that various Euler operators proposed in the literature for 2D and 3D cell complexes are either instances of our operators, or can be expressed in terms of them.

Virtually all proposed sets of basis Euler operators contain *MEV* (Make Edge and Vertex) and *MEF* (Make Edge and Face) operators, which are the same as our *M0C1C* (Make 0-cell and 1-cell) and *M1C2C* (Make 1-cell and 2-cell) operators, respectively.

Several sets of basis operators have been proposed for manifold 2-complexes bounding a solid in $\mathbb{E}^3$, called *boundary models*. In these models, there is only one implicitly represented volumetric cell (corresponding to the cavity determined by the complex), which is not necessarily a topological cell. Various forms of topology-modifying operators [EW79, BHS80, MS82, Man88] are defined for such models.

The *glue* operator in [EW79] merges two faces and deletes both of them. It corresponds to the connected sum operator on manifold surfaces. Two faces may be glued if they have the same number of vertices, and they have no edges in common. The glue operator deletes not only the two faces, but it deletes also all the edges and vertices on the boundary of one of the deleted faces. There are two instances of the glue operator, illustrated in Figure 2.

- if the two glued faces belong to two different shells, one shell is deleted (β_0 is decreased by one), and the operator is called *KFS* (Kill Face and Shell).
- if the two glued faces belong to the same shell, a handle (genus) is created (β_1 is increased by two), and the operator is called *KFMH* (Kill Face, Make Hole).

Fig. 2. Two instances of the glue operator: *KFS* (Kill Face and Shell) (a), *KFMH* (Kill Face, Make Hole) (b)

Let $v_1, e_1, v_2, e_2, \dots, v_k, e_k$ and $v'_1, e'_1, v'_2, e'_2, \dots, v'_k, e'_k$ be the cyclical lists of vertices and edges of the two glued faces f and f' , listed in the order in which they are identified. Both instances of the glue operator can be expressed through our operators as follows:

- *M1CK0Cycle* for *KFS* and *M1C1Cycle* for *KFMH* creates an edge connecting v_1 and v'_1 ,
- *K0C1C* contracts the edge (v_1, v'_1) , and identifies v'_1 with v_1 (vertex v_1 is the current vertex),
- *M1C2C* makes a triangular face with vertices v_i, v_{i+1}, v'_{i+1} for current vertex v_i , *K0C1C* and *K1C2C* identify vertex v'_{i+1} with vertex v_{i+1} and edge e'_{i+1} with edge e_{i+1} , respectively (v_{i+1} is the current vertex),
- *M2C2Cycle* and *K1C2C* identify edge e'_k with edge e_k ,
- *K2C2Cycle* deletes face f' and the 2-cycle formed by faces f and f' ,
- *K2C2Cycle* for *KFS* merges the two solids bounded by shells containing f and f' into one, and *K2CM1Cycle* for *KFMH* deletes face f and creates a 1-cycle determined by edges $e_1, \dots, e_k$.

In [BHS80, MS82, Man88], the topology-modifying operator is called *MRKF* (*Make Ring, Kill Face*). It is similar to the glue operator in [EW79], but it imposes less restrictive conditions on the glued faces, and it deletes only one of the faces. It creates a ring and deletes a face from the model, by gluing a face to another face, thus deleting one face and making an (inner) ring in another face. The face that is not deleted is transformed into a non topological cell. The operator has two instances:

- *MRKFS* (*Make Ring, Kill Face and Shell*) glues together two faces belonging to two different shells, thus merging two shells into one.
- *KFMRH* (*Kill Face, Make Ring and Hole*) glues two faces belonging to the same shell, thus making a hole (genus) in the surface.

Let f' be the face that is glued to face f , and deleted. *MRKF* can be expressed through our operators as follows (see Figure 3):

- k *M0C1C* operators, where k is the number of edges and of vertices of f' ,
- k *M1C2C* operators, which create a copy of f' in f ,
- a sequence defining the glue operator in [EW79],
- $(k - 1)$ *K1C2C* operators (we leave one edge joining a vertex of f to a vertex of f' to maintain the topological validity of face f).

Topology-modifying operators defined for non-manifold 2-complexes in $\mathbb{E}^3$ [LL01] are called *MECh* (*Make Edge and Complex Hole*), *MFKCh* (*Make Face, Kill Complex Hole*) and *MFCc* (*Make Face and Complex Cavity*). They are the same as our operators *M1C1Cycle*, *M2CK1Cycle* and *M2C2Cycle*, respectively. For 3-complexes in $\mathbb{E}^3$ [MSNK89, Mas93], an additional topology-modifying operator is defined, called *MVlKc* (*Make Volume, Kill Complex Cavity*), which is the same as our *M3CK2Cycle* operator.

Fig. 3. *MRKF* operator expressed through our operators: triangular face to be glued to the quadrangular face (a), three *M0C1C* operators (b), three *M1C2C* operators (c), *glue* (d), two *K1C2C* operators (e)

5 Comparison with Handle Operators

Handle operators on a manifold cell 2-complex Γ triangulating a surface S have been introduced in [LPT⁺03]. They are based on handlebody theory for surfaces [Mil63, Mat02], stating that any surface S can be obtained from a 2-ball by iteratively attaching handles (0-, 1- and 2-handles).

Attachment of a 0-handle is also an initialization operator. It creates a new surface with one face, three edges and three vertices. There are three operators that correspond to attaching a 1-handle. They identify two boundary edges of Γ (incident in exactly one face) with no vertices in common. If the two identified edges belong to two different components of Γ , then the number of connected components and of boundary curves (connected components of boundary edges) in Γ is decreased by one. If the two identified edges belong to the same component and the same boundary curve of Γ , then the number of holes (independent 1-cycles) and the number of boundary curves in Γ is increased by 1. If the two identified edges belong to the same component and two different boundary curves of Γ , then the number of holes (independent 1-cycles) is increased by 1, and number of boundary curves in Γ is decreased by 1. The operator that corresponds to the attachment of a 2-handle identifies two edges on the boundary of Γ with two vertices in common. It decreases the number of holes and the number of boundary curves in Γ by 1. Handle operators in 2D are illustrated in Figure 4.

Fig. 4. Handle operators in 2D: attachment of a 0-handle (a); attachment of a 1-handle (b), (c) and (d); attachment of a 2-handle (e)

Handle operators can be classified as topology-modifying operators, and they can be expressed through our operators as discussed below:

- The attachment of a 0-handle corresponds to creating an initial triangle (a 2-ball). It can be expressed as *M0C0Cycle* operator, two *M0C1C* operators and one *M1C2C* operator, which together create a triangle (see Figure 5 (a)).

- The attachment of a 1-handle identifies two boundary edges e_1 and e_2 with no vertices in common. It can be expressed through one $M1CK0Cycle$ and one $M1C1Cycle$ operator if e_1 and e_2 belong to different components, or two $M1C1Cycle$ operators if they belong to the same component (the created edges connect the endpoints of e_1 to the corresponding endpoints of e_2), two $K0C1C$ operators (they contract the two created edges and identify the corresponding endpoints), one $M2CK1Cycle$ operator (it creates a face that fills the ring and deletes the cycle formed by e_1 and e_2), and finally one $K1C2C$ operator (it contracts the created face and identifies e_1 with e_2) (see Figure 5 (b)).
- The attachment of a 2-handle identifies two edges with both vertices in common. It can be expressed as a $M2CK1Cycle$ operator, followed by $K1C2C$ operator.

Fig. 5. Attachment of a 0-handle in 2D can be expressed as one $M0C0Cycle$, two $M0C1C$ and one $M1C2C$ (a). Attachment of a 1-handle in 2D can be expressed as one $M1CK0Cycle$ or one $M1C1Cycle$, one $M1C1Cycle$, two $K0C1C$, one $M2CK1Cycle$ and one $K1C2C$ (b).

Handle operators have been extended to 3D in [LT97]. The operator that creates a new 3-ball (initialization operator) corresponds to the attachment of a 0-handle. Other operators identify two boundary faces (incident in exactly one 3-cell) of a cell 3-complex Γ triangulating a solid S . The attachment of a 1-handle can be applied in three situations: if the two identified boundary faces are on different connected components of Γ , then the two components are merged into one; if the two identified faces belong to the same boundary surface component of Γ (connected component of boundary faces) and have no edges in common, then a hole is created; if the two identified faces belong to the different boundary surfaces of the same connected component of Γ , the operator can be realized only if Γ is embedded in a space of dimension greater than 3. The attachment of a 2-handle corresponds to identifying two faces on the same boundary surface component of Γ that have some edges in common. The operator can create cavities and/or close holes in Γ . The attachment of a 3-handle is applicable if the two identified faces belong to the same boundary surface component and have all edges in common. This operator fills in the cavity formed by the two identified faces.

The handle operators in 3D generalize the glue operator in [EW79], since the two faces identified by a handle operator may have none, some, or all edges in common. Thus, they can be expressed in terms of our operators in a similar manner.

6 Concluding Remarks

We have introduced a complete set of basis operators for building and updating cell complexes in arbitrary dimensions. We have shown how other Euler operators and handle operators proposed in the literature on 2D and 3D cell complexes can be expressed through our operators.

An interesting research direction would be to generalize handle operators to higher dimensions, i.e., for an n -manifold discretized as a cell complex, and to express them as macro-operators through Euler operators in higher dimensions.

Simplicial complexes, which can be considered as a special case of cell complexes, have been used as a preferred representation of shapes long before cell complexes. In [LLM⁺10], a unified framework for building and updating manifold cell 2-complexes has been proposed. It combines topology-preserving stellar operators, and topology-modifying handle operators. We believe that it would be interesting to combine our topology-modifying operators with stellar operators.

In [Gom04], the operators in [MSNK89] have been extended to complexes called *stratifications*, in which cells, called *strata*, are defined by analytic equalities and inequalities. The cells are not necessarily homeomorphic to a ball, and they may have incomplete boundaries. We plan to investigate the relationship between our operators and the ones in [Gom04], in the case of cell complexes.

Hierarchical pyramidal models have been defined in the framework of combinatorial maps [BK03], based on topology-preserving operators. We plan to define a multi-resolution model for cell complexes based on our operators.

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