Computing Morse Decompositions for Triangulated Terrains: An Analysis and an Experimental Evaluation

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Abstract. We consider the problem of extracting the morphology of a terrain discretized as a triangle mesh. We discuss first how to transpose Morse theory to the discrete case in order to describe the morphology of triangulated terrains. We review algorithms for computing Morse decompositions, that we have adapted and implemented for triangulated terrains. We compare the the Morse decompositions produced by them, by considering two different metrics.

1 Introduction

Modeling the morphology of a real terrain is a relevant issue in several application domains, including terrain analysis and understanding, knowledge-based reasoning, and hydrological simulation. Morphology consists of feature points (pits, peaks and passes), feature lines (like ridges and ravines), or segmentation of the terrain in regions of influence of minima and maxima or in regions of uniform gradient field. Morphological models of terrains are rooted in Morse theory [26]. Based on Morse theory cellular decompositions of the graph of a scalar field defined on a manifold, called *Morse complexes*, are defined by considering the critical points of the field and the integral lines. The same mathematical and algorithmic tools apply for segmenting 3D shapes based on a scalar field defined over the shape, such as curvature [22,23,24]. Unfortunately, the results from Morse theory apply only to smooth functions, while the most common terrain and shape models are piece-wise linear functions defined over a triangle mesh. Current approaches [4,6] try to simulate Morse complexes in the discrete case, but pose constraints (i.e., flat edges are not allowed), that are usually not satisfied by real terrains or 3D shapes.

The purpose of this work is to review, analyze and compare algorithms for computing Morse complexes on triangulated surfaces built on real data sets. We focus our attention on terrains modeled as piece-wise linear triangulated surfaces with vertices at the sample points, called *Triangulated Irregular Networks* (*TINs*). First, we discuss how to overcome the limitations of current discrete approaches by dealing with flat edges effectively. We classify flat edges and then discuss how to eliminate singularities from TINs through a process of edge collapse which may generate new critical points.

In the literature, two major approaches exist for computing Morse decompositions of discrete terrain models: *region-based* algorithms compute the 2-cells of the Morse complexes while *boundary-based* algorithms compute the lines forming their boundaries. We compare results on real datasets, and we consider both noisy data and data after a smoothing process and elimination of the singularities. We compare the segmentations produced by the various algorithms using two metrics.

The remainder of the paper is organized as follows. In Section 2, we present background notions on Morse theory. In Section 3, we present discrete approaches to Morse theory and our new characterization of Morse functions in the discrete, that we call *extended Piecewise Linear (ePL)-Morse* functions. In Section 4, we review algorithms to decompose a terrain into Morse complexes. In Section 5 we discuss the metrics, used in Section 6 to compare the Morse decompositions produced by the different algorithms. Finally, in Section 7, we draw some concluding remarks, and discuss future developments.

2 Background Notions

Let us consider a domain M in the three-dimensional Euclidean space and a smooth real-valued function f defined over M . A point p of M is a *critical point* of f if the gradient of f at p is null. Critical points of f are *minima*, *saddles* and *maxima*. All other points are called *regular*. A critical point p is *degenerate* if the determinant of the Hessian matrix of f at p is zero. A function f is called a *Morse function* [26], if all its critical points are not degenerate. A Morse function admits only finitely many isolated critical points.

An *integral line* of f is a maximal path which is everywhere tangent to the gradient of f . Integral lines that converge to a minimum [saddle, maximum] p form a 0-cell [1-cell, 2-cell] called the *stable cell* of p . Similarly, integral lines that originate from a minimum [saddles, maximum] p form a 2-cell [1-cell, 0-cell] called the *unstable cell* of p . The stable (unstable) cells decompose M into a Euclidean cell complex, called a *stable (unstable) Morse complex*. If the unstable and the stable cells intersect transversally, function f is said to satisfy the *Smale condition*. If f satisfies the Smale condition, the overlay of stable and unstable Morse complexes is called a *Morse-Smale complex*.

A Morse complex is also related with the concept of watershed transform [29], which provides a decomposition of the domain of a smooth function f into open regions of influence associated with the minima of f , called *catchment basins*. Catchment basins correspond to the cells of the stable Morse complex defined on the same domain. The regions of influence related to the saddles form the *watershed lines*.

3 Discrete Approaches to Morse Theory

In the literature there are two extensions of Morse Theory to a discrete domain. The *discrete Morse theory* proposed by Forman [18] presents an adaptation of

Morse theory that can be applied to any simplicial or cell complex. Forman assumes to have a function value associated with each cell of the complex (not just the vertices). Thus, it is not directly useful in our case. The *piecewise linear Morse theory* by Banchoff [4] extends Morse theory to piecewise-linear functions defined on triangulated surfaces, under the assumption is that every pair of points of the triangulated surface have distinct field values. In order to define the conditions for a vertex v to be critical, the polyhedral surface made by the triangles incident in v is considered, and the number I of intersections between such surface and the plane parallel to the x - y -plane, which is passing through v , is counted. If there are no intersections, v is a maximum or a minimum. If there are two intersections, v is a regular vertex. Otherwise v is a saddle: a *simple saddle* if $I = 4$, or a *multiple saddle* if $I > 4$. Note that I is always even. The *multiplicity* of a saddle is equal $I/2$.

Banchoff's assumption on piece-wise linear Morse functions is quite strong, and thus it is often replaced by the weaker condition that each pair of adjacent vertices (i.e., connected by an edge) have different elevation values [6]. Such condition ensures a decomposition of the domain, which has the same properties as a Morse complex, called a *Piece-wise Linear Morse Complex (PLMC)*. The above condition is still too strong for real data sets. In [15], we have defined a classification of an isolated flat edge e based on the value f_e of function f on e , and on the values f_C and f_D of f at the other vertices of the triangles incident into e . If $f_C < f_e < f_D$ or $f_D < f_e < f_C$, then e is a *regular edge*. If $f_e < \min(f_C, f_D)$ or $f_e > \max(f_C, f_D)$ then e is a *critical edge*. We say that the elevation function defined at the vertices of a TIN is an *extended PL (ePL)-Morse function* if it does not contain critical edges or flat triangles. TINs that present only isolated flat edges can be reduced to TINs for ePL-Morse functions by collapsing such edges.

4 Algorithms for Computing Morse Decompositions

In the literature several approaches have been proposed for providing an approximation of the Morse and Morse-Smale complexes in the discrete.

Boundary-based algorithms (see Section 4.1) extract the Morse-Smale complex by tracing the integral lines, or their approximations, starting from saddle points and converging to minima and maxima. *Region based* algorithms compute the 2-cells of the Morse complexes by growing the neighborhood of minima and maxima (see Section 4.2). *Watershed* algorithms compute the Morse complexes based on the discrete *watershed transform* (see Section 4.3).

4.1 Boundary-Based Algorithms

Boundary-based algorithms [31,3,8,17,27,31] compute the 2-cells of the Morse-Smale complex on a TIN indirectly by computing the 1-cells, i.e., the boundaries of the 2-cells. They first extract the critical points, and then trace the separatrix lines, that are the 1-cells of the Morse-Smale complex.

The basic idea is due to Takahashi et al [31]. According to Banchoff [4], critical vertices are identified by considering the incident triangles in them. Let us consider the radially sorted list of neighbors of a saddle s . Within such sorted list, we consider the maximal sublists formed by vertices with higher (lower) value than s and call them *upper (lower) sequences*. Starting from each saddle p , with multiplicity k , $2k$ path are computed. Given p , we consider the upper sequences and the lower sequences. We select the first point of each path by choosing the highest (lowest) vertex from each upper (lower) sequence. Then, the ascending (descending) paths are computed by choosing, at each step, the highest (lowest) adjacent vertex until a critical point is reached. Saddles play a key role in this algorithm.

Edelsbrunner et al. [17,16] extend the approach by Takahashi et al. [31]. Function f is required to be a piecewise linear Morse function, according to Banchoff's definition, i.e., vertices must have distinct function values. Likewise, ridges and valleys are computed by starting from saddles, but at each step the point with the maximum slope is selected. The Smale condition is simulated a-posteriori by extending the path beyond each saddle, and forcing the path of the critical net not to intersect. The technique is rather complex, it requires edge duplications, and can lead to degenerated 2-cells (see [17] for more details).

4.2 Region-Growing Algorithms

Region growing algorithms [9,12,20] mimic, in the discrete case, the definition of 2-cell for a Morse complex, which is the set of the integral lines that originate or converge to a critical point. We discuss here how the algorithms extract the unstable Morse complex. The computation of the stable one is done in a completely symmetric way.

We review here three region growing methods proposed in our previous work. The algorithm in [12], denoted here as *DIS*, is based on the elevation values at the TIN vertices, while the one in [11], denoted as *GRD*, is based on the gradient associated with the triangles. The algorithms sort the vertices according to their elevation and process them in decreasing elevation order. Let p be p the vertex at the highest elevation among unprocessed vertices. The unstable 2-cell associated with p , that we denote with γ , is initialized with all the triangles incident in p , which have not yet been assigned to any 2-cell. The 2-cell γ is grown by following a predefined criterion until it cannot be extended any more. If the vertex p lies on the boundary of another 2-cell γ' , then p is not a local maximum, and thus γ and γ' are merged.

The region growing criterion, according to which a triangle is added to the 2-cell γ , is different in the two algorithms. In the *DIS* algorithm, the current 2-cell γ is extended by including the triangle $t = ABC$, adjacent to γ along an edge AB , if C is the vertex f t with lowest elevation. Instead, the *GRD* algorithm includes t if the gradient of t has the same orientation as the gradient of t' , where $t' = ABD$ is the triangle in γ sharing edge AB with t .

In the growing algorithm in [20], denoted here as *STD*, the region growing criterion plays the most important role. Initially, the algorithm labels the highest,

middle, and lowest vertex in each triangle t as *source* (S), *through* (T), and *drain* (D), respectively. Then, the minima are found as those vertices labeled D in all their incident triangles. For each minimum m , the 2-cell γ associated with m is initialized with all triangles incident in m . A loop follows, in which, at each iteration, the algorithm selects a set of triangles to be included from an edge e of the boundary of the current 2-cell γ , based on vertex classification.

4.3 Watershed Algorithms

The watershed transform has been first introduced in image processing for gray-scale images. Several definitions, and related algorithms, exist in the discrete case [5,7,21,25,30,32] to compute the watershed transform (see [29] for a survey). Here, we extended and implemented the two major approaches proposed in the literature, based on simulated immersion [32], and on the discretization of the topographic distance [25], to deal with TINs, and in general triangle meshes in 3D space having a scalar value associated with each vertex. We denote the simulated immersion algorithm for TINs as *WVS* and as the one using topographic distance as *WTD*.

5 Metrics for Comparison

In this Section, we describe the metrics we use to compare the decompositions of the same TIN produced by different algorithms. Such metrics provide a number between 0 and 1, where 0 indicates that two decompositions Γ_1 and Γ_2 are completely different, and 1 indicates that they are identical [10].

The first metric, called *Rand index (RI)* [28,10], measures the likelihood that a pair of triangles is either in the same region or in different regions in both decompositions. For each pair of triangles, we mark 1 either if they belong to one region in both decompositions, or if they belong to two different regions in both decomposition. Otherwise, we mark 0. Then, we divide the sum of the values by the number triangle pairs. *RI* metric assesses the similarity among decompositions even in case in which number of regions is widely different and region matching is problematic.

The second metric is a variant of the Hamming distance defined in [19,10]. The general idea is to find a best corresponding 2-cell in decomposition Γ_1 for each 2-cell in decomposition Γ_2 , and measure the difference area between the regions. In the original definition, region matching is performed by considering the maximum common area. In our case, we say that two regions (2-cells) are corresponding if they have the same critical point associated. We define as *Hamming distance (HD)* the ratio between the number of the triangles that are assigned to corresponding 2-cells in the two decompositions, and the total number of triangles. The *HD* metric heavily relies on region matching, and is strongly affected even in case of a few mismatches.

6 Experimental Results

We have applied the algorithms presented in Section 4 both on synthetic and real datasets, and compared the Morse decompositions by using the metrics described in Section 5. Real data are affected by sampling errors, and, in most cases, contain flat areas (plains, lakes etc.); thus, real TINs do not correspond to Morse functions.

6.1 Input Data

We have experimented on three groups of real terrain data sets. The datasets of the first group are TINs over subsets of data from the CGIAR-CSI GeoPortal, [1]. Such TINs are composed of vertices distributed on a regular square grid and isosceles right triangles all of the same size. In the following, we refer to them as *regular TINs*. These TINs are available on the Aim@Shape repository [2]. The TINs of second group are extracted from a regular data set but triangulated in a nested triangle structure, called a *hierarchy of diamonds* [34]. The resulting TINs are crack-free triangle meshes extracted from the hierarchy and are composed by isosceles right triangles of different sizes. In the following, we refer to them as *diamonds TINs*. The third group contains irregular triangle meshes, that we call *irregular TINs*.

Table 1. Datasets used in the experiments

	Name	#Vertices	#Triangles	ePL
Regular	MontBlanc	14 400	28 322	reducible
	MontBlanc-ePL	14 394	28 310	yes
	Elba	36 036	70 220	reducible
	Elba-ePL	35 365	69 059	yes
	Monviso	160 000	318 402	reducible
	Monviso-ePL	159 606	317 623	yes
Diamonds	San Bern.	8 022	15 610	reducible
	San Bern-ePL	7 854	15 286	yes
	Marcy0003	214 287	427 242	no
	Marcy003	14 089	27 502	reducible
	Marcy003-ePL	13 722	26 867	yes
	Hawaii0003	120 456	240 840	no
	Hawaii001	15 196	29 599	no
Irregular	Vobbia	6 095	11 838	yes
	Zqn	57 865	114 765	reducible
	Zqn-ePL	57 816	114 695	yes
	Dolomiti	10 004	19 800	reducible
	Dolomiti-ePL	9 983	19 759	yes
	Marcy50	50 004	99 345	no

Fig. 1. Unstable decompositions of Vobbia, produced by TKH (a) and BBD (b)

Fig. 2. Comparison between stable decompositions of Monviso (a) and Monviso ePL (b), produced by WVS and WTD. Red areas represent mismatching

We classify those meshes into (i) ePL-Morse meshes, i.e., meshes that have no critical flat edges, (ii) meshes which can be reduced to ePL-Morse meshes, and (iii) meshes not reducible to ePL-Morse ones. For meshes that are reducible to ePL-Morse meshes, we consider both the original mesh and the mesh obtained by replacing each critical flat component with a component that is not flat, but has the same critical features. We denote those meshes with *-ePL* suffix. Table 1 shows the list of our test TINs. Figure 1 shows unstable decompositions of Vobbia TIN produced by TKH (a) and BBD (b).

6.2 Results

On ePL-meshes, all algorithms find the same number of regions in both the stable and the unstable decompositions. We have compared the algorithms pairwise, according to the Rand-Index (RI) and Hamming-Distance (HD) metrics. RI has, for each pair of algorithms, very high values, almost always in the top 2%. For regular or diamonds ePL-meshes, HD is in the range 80%–99%. In particular, TKH, WVS and STD algorithm give very similar decompositions (HD is higher than 95%). As soon as the shape of triangles become less regular, decompositions computed by TKH, WVS and STD algorithms start diverging, and HD becomes as low as 50%. GRD, WTD and BBD, that simulate the differentiability by considering in some way the slope of triangles, give always pretty similar results, with HD from 80% to 90%.

Decompositions obtained from non-ePL-meshes can be quite different, even if the number of critical edges is small. Figure 2 shows mismatching between WVS and WTD for a non-ePL-meshe and its ePL version. When we have a

large number of critical edges, or wide flat areas, the mesh cannot be reduced to ePL-Morse, and both metrics show very low values. For the Hawaii dataset RI is in the range 57-82%, and HD is in the range 21-43%, mainly due to the fact that the dataset is an island, and the boundary is a minimum composed by a long chain of edges. On Marcy dataset, RI is the range 89-95%, and HD is in the range 20-42%, due to a large flat area representing a river with small islands inside. For full details on the metrics comparing decomposition algorithm pairwise, please refer to [33].

7 Concluding Remarks

We have considered the problem of computing the morphology of triangulated terrains and we have presented and analyzed the various approaches for computing it. In Table 2, we summarize the main properties of the algorithms. GRD and WVS sort vertices and detect maxima and minima during computation, while STD, WTD, TKH and BBD use Banchoff’s characterization (see Section 3) to find maxima and minima in a preprocessing step. STD, WVS and TKH are fully discrete and consider only the elevation values at the vertices. Decompositions of regular grid are similar for all fully discrete algorithms. GRD, WTD and BBD, take edge slope into account, but simulate the differentiability in various ways. Decompositions resulting from regular TINs are quite similar. Resulting decompositions on irregular TINs are different, but GRD, WTD and BBD that simulate the differentiability (even if in different ways), give results more similar among them than STD, WVS and TKH algorithms.

GRD and WVS can deal with higher dimensional data. WTD can be extended quite easily, while STD in higher dimension requires handling of a large number of local configurations. BBS and TKH are not extensible to higher dimensions, since they compute separatrix lines, while in higher dimensions we should compute hypersurfaces. See [33] for further details.

Table 2. Main properties of the algorithms

	Extraction of critical points	Simulation of Differentiability	Scale easily to higher dimension
GRD	Sort vertices	Gradient	Yes
STD	Banchoff	Fully Discrete	No
WTD	Banchoff	Top. Distance	No
WVS	Sort vertices	Fully Discrete	Yes
TKH	Sort vertices	Fully Discrete	No
BBD	Sort vertices	Diff. Quotient	No

The size of a morphological representation can be large for common models, which may consist of several millions of triangles. To deal with problem, we have been focusing on a multiresolution representation of the terrain morphology (see [13]), that allows concentrating on the areas of interest and producing approximate representations of the morphology at uniform and variable resolution. In our current work, we are developing a multiresolution model that combines the geometrical and morphological representations, where the morphological representation as an index on the geometrical model [14].

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