Chapter 31
Unbending of Curved Tube by Internal Pressure

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Abstract In this work the effect of the unbending of a curved tube under a uniform normal pressure is investigated. The problem is considered within the framework of the nonlinear membrane theory. It is shown that the inflation of a curved tube is the special case of pure bending. The tube with a circular cross section made of a Mooney-Rivlin material is studied numerically. The dependencies between the curvature of the centerline of deformed curved tube and the internal pressure are obtained. It is found that there are the maximum pressures for the considered materials.

Keywords Nonlinear elasticity · Membrane · Curved tube

31.1 Introduction

A membrane that is a sector of a torus is called a curved tube. The problem of pure bending of pressurized curved tubes is considered within the framework of the nonlinear shell theory by Libai and Simmonds [5] and by Zubov [7]. In [5, 7] the approach is proposed to solve the problem of pure bending. The approach allows us to decompose the deformation into two parts: an in-plane deformation of meridional cross section, plus a rigid rotation of each of these meridional planes about some axis by linearly varying angles. In this case the equilibrium equations are reduced to the ordinary differential equations.

This approach is used to solve the pure bending problem of a straight tube (cylindrical membrane) [4, 7]. The change of curvature is the result of the application of the bending moments to a straight tube. The feature of curved tube is the unbending

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under internal pressure in the absence of bending moments. This phenomenon is investigated in the presented work.

In Sect. 31.2 the problem of pure bending is considered. Using the semi-inverse method the system of the ordinary differential equations are derived. It is shown that the inflation of a curved tube is the special case of a pure bending. In Sect. 31.3 we consider the inflation of a straight tube.

The aim of this investigation is to produce curves of dimensionless pressure versus dimensionless curvature of the tube. The results are presented for the tube of a circular cross section made of a Mooney–Rivlin material in Sect. 31.4. The inflation of curved tube is compared with the inflation of straight tube. It is obtained that there are the maximum pressures for the considered materials. The maximum pressures for straight and curved tubes are closely.

31.2 Pure Bending Deformation

Denote by o the surface of membrane in the reference configuration. The position of a point on o is given by

$$\mathbf{r} = \chi_1(q^1)\mathbf{i}_1 + \chi_2(q^1)\mathbf{e}_2, \quad q^1 \in [q^1_1; q^1_2], \quad q^2 \in [q^2_1; q^2_2], \qquad (31.1)$$
$$\mathbf{e}_2 = \mathbf{i}_2 \sin\beta q^2 + \mathbf{i}_3 \cos\beta q^2.$$

Here q^1 and q^2 are the Gaussian surface coordinates, the basis $\{\mathbf{i}_k\}$ $(k = 1,2,3)$ is the Cartesian basis. We assume that the cross section ($q^2 = const$) is a closed and it is given by the functions $\chi_1(q^1)$ and $\chi_2(q^1)$. The parameter β is the curvature of the centerline of the torus sector. It is called the initial curvature of curved tube.

We assume that the deformed membrane is a sector of torus. Denote by O the surface of deformed membrane. The position of a point on O is given by

$$\mathbf{R} = \mathrm{X}_1(q^1)\mathbf{i}_1 + \mathrm{X}_2(q^1)\mathbf{E}_2, \quad \mathbf{E}_2 = \mathbf{i}_2 \sin \mathrm{B} q^2 + \mathbf{i}_3 \cos \mathrm{B} q^2. \qquad (31.2)$$

Here the deformed cross section is given by the unknown function $\mathrm{X}_1(q^1)$ and $\mathrm{X}_2(q^1)$. The unknown parameter B is the curvature of the deformed centerline. It is called the curvature of deformed curved tube.

The covariant components of the metric tensors of o and O are independent of the Gaussian coordinate q^2 and they form the diagonal matrices [4]

$$g_{11} = {\chi_1'}^2 + {\chi_2'}^2, \quad g_{12} = 0, \quad g_{22} = \beta^2\chi_2^2, \qquad (31.3)$$

$$G_{11} = {\mathrm{X}_1'}^2 + {\mathrm{X}_2'}^2, \quad G_{12} = 0, \quad G_{22} = \mathrm{B}^2\mathrm{X}_2^2, \qquad (31.4)$$

$$B_{11} = \frac{\mathrm{X}_1''\mathrm{X}_2' - \mathrm{X}_1'\mathrm{X}_2''}{\sqrt{{\mathrm{X}_1'}^2 + {\mathrm{X}_2'}^2}}, \quad B_{12} = 0, \quad B_{22} = \frac{\mathrm{B}^2\mathrm{X}_2\mathrm{X}_1'}{\sqrt{{\mathrm{X}_1'}^2 + {\mathrm{X}_2'}^2}}. \qquad (31.5)$$

Here the prime denotes differentiation with respect to q^1, $g_{\alpha\gamma}$ $(\alpha,\gamma = 1,2)$ are the covariant components of the first metric tensor of o, $G_{\alpha\gamma}$ are the covariant components of the first metric tensor of O and $B_{\alpha\gamma}$ are the covariant components of the second metric tensor of O.

If the thickness h of the membrane is a constant and the conditions (31.3)–(31.5) are satisfied then the deformation is called the one-dimensional deformation. In this case the equilibrium equations reduce to the system of ordinary equilibrium equations [4]. The conditions (31.3)–(31.5) put some restrictions on an external surface load. It is independent of q^2 and its component is equal zero along the coordinate lines q^2. We assume that the external surface load is constant normal pressure p. The equilibrium equations may be reduced to the form [4]

$$\frac{dL^{11}}{dq^1} + L^{11}\left(2\Gamma^1_{11} + \Gamma^2_{21}\right) + L^{22}\Gamma^1_{22} = 0, \tag{31.6}$$

$$L^{11}B_{11} + L^{22}B_{22} + p = 0. \tag{31.7}$$

Here $L^{\alpha\gamma}$ are the components of the Cauchy stress resultant tensor [4, 6].

We introduce the principal stretches $\lambda_1(q^1)$, $\lambda_2(q^1)$ and the function $\psi(q^1)$

$$\lambda_1(q^1) = \sqrt{\frac{G_{11}}{g_{11}}}, \quad \lambda_2(q^1) = \sqrt{\frac{G_{22}}{g_{22}}}, \quad \tan\psi(q^1) = \frac{\mathrm{X}_2'(q^1)}{\mathrm{X}_1'(q^1)}. \tag{31.8}$$

Denote by W the strain energy density of the membrane. The strain energy density can be expressed as function of the principal stretches for incompressible isotropic elastic material: $W = W(\lambda_1,\lambda_2)$. Therefore, using (31.4) and (31.8), the constitutive relations may be written in the form [4]

$$L^{11} = \frac{h}{g_{11}\lambda_1^2\lambda_2}\frac{\partial W}{\partial\lambda_1}, \quad L^{22} = \frac{h}{g_{22}\lambda_1\lambda_2^2}\frac{\partial W}{\partial\lambda_2}, \quad L^{12} = L^{21} = 0. \tag{31.9}$$

Finally, using (31.8) and the constitutive relations (31.9), the equilibrium equations (31.6) and (31.7) reduce to the following system

$$\frac{\partial^2 W}{\partial\lambda_1^2}\lambda_1' - \left(\frac{\partial W}{\partial\lambda_2} - \lambda_1\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}\right)\sqrt{\frac{g_{11}}{g_{22}}}\mathrm{B}\sin\psi + \\ + \left(\frac{\partial W}{\partial\lambda_1} - \lambda_2\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}\right)\frac{g_{22}'}{2g_{22}} = 0, \tag{31.10}$$

$$\lambda_2' - \mathrm{B}\sqrt{\frac{g_{11}}{g_{22}}}\lambda_1\sin\psi + \frac{g_{22}'}{2g_{22}}\lambda_2 = 0, \tag{31.11}$$

$$\frac{\partial W}{\partial\lambda_1}\psi' - \mathrm{B}\sqrt{\frac{g_{11}}{g_{22}}}\lambda_1\cos\psi - \frac{p}{h}\sqrt{g_{22}}\lambda_1\lambda_2 = 0, \tag{31.12}$$

$$\mathrm{X}_1' = \sqrt{g_{11}}\lambda_1\cos\psi, \tag{31.13}$$

$$\mathrm{X}_2' = \sqrt{g_{11}}\lambda_1\sin\psi. \tag{31.14}$$

If the parameter B is fixed then the Eqs (31.10)–(31.14) and periodicity conditions are the boundary-value problem to determine the unknown functions $X_1(q^1)$ and $X_2(q^1)$.

A solution can be satisfied integral edge conditions at the edge of torus sector ($q^2 = q^2_1$ and $q^2 = q^2_2$) [5, 7]. It can be shown that the resultant force **F** and the resultant moment **M** are independent of the cross section, $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = M_1(q^1)\mathbf{i}_1$ [4]. Denote by Y_{2C} the coordinate of the center mass of surface which is bounded by the membrane at cross section. Then we have [4]

$$M_1 = \int_{q^1_1}^{q^1_2} \sqrt{G_{11}} G_{22} L^{22} (Y_{2C} - X_2)\, dq^1.$$

The considered deformation is the pure bending deformation. If the membrane is loaded by the uniformly normal pressure then $M_1 = 0$. From this condition we may determine the curvature B of the deformed torus sector.

Thus, the problem of the unbending a curved tube by an internal pressure reduces to the boundary-value problem with the parameter B. The boundary-value problem is solved numerically using a shooting method together with a Runge-Kutta integration process. The parameter B is determined by the condition $M_1 = 0$ using shooting method.

31.3 Cylindrical Membrane

We now consider the limiting case when the membrane is a cylinder (straight tube) with a closed cross section in the reference configuration . The external load is the uniformly normal pressure p. We use the integral edge conditions $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$

A closed cross section of the undeformed cylindrical membrane may be given as a circle of radius r_0 [2]. If a cylindrical membrane is subjected by the uniformly distributed pressure only then the deformed membrane is the cylinder with a circle cross section of radius R_0 [1, 2]

$$R_0 = \frac{L^{11} G_{11}}{p}. \tag{31.15}$$

It follows from symmetry that the consition $\mathbf{M} = \mathbf{0}$ is satisfied. The resultant force at the edge is represented as the sum of the pressure p and the stress L^{22}. Hence we have

$$R_0^2 \pi p - L^{22} G_{22} 2 R_0 \pi = 0. \tag{31.16}$$

Using the constitutive relations (31.9) and

$$\lambda_1 = \frac{R_0}{r_0}, \quad \lambda_2 = \gamma,$$

the Eqs (31.15) and (31.16) may be rewritten in the form

$$p = \frac{r_0 h}{\lambda_1 \lambda_2} \frac{\partial W}{\partial \lambda_1}, \quad p = \frac{2 r_0 h}{\lambda_1^2} \frac{\partial W}{\partial \lambda_2}. \tag{31.17}$$

Here the unknown parameter γ determines the change of the cylinder length.

If we shall give the strain energy density W and the pressure p then the unknown geometric parameters R_0 and γ of the deformed cylindrical membrane are determined from the Eqs (31.17).

31.4 Circular Cross Section

Let the cross section of undeformed curved tube is the circle of radius r_0

$$\chi_1(q^1) = r_0 \sin q^1, \chi_2(q^1) = \beta^{-1} - r_0 \cos q^1, \quad q^1 \in [0; 2\pi], q^2 \in \left[0; \frac{\pi}{2\beta}\right].$$

In this work we consider the three initial curvature $\beta = 0.05, 0.1, 0.2$.

Consider a Mooney–Rivlin material. The strain energy density may be written in the form

$$W = \frac{\mu}{4}\left[(1+\nu)\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3\right) + (1-\nu)\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2 \lambda_2^2 - 3\right)\right].$$

We assume that

$$r_0 = 1, \quad h = 0.001, \quad \mu = 1.$$

The numeric results are obtained for $\nu = 1$ and $\nu = 0.5$. In case $\nu = 1$ a Mooney–Rivlin material is also called a Neo–Hookean material.

We introduce the dimensionless parameters

$$p^* = \frac{p r_0}{\mu h}, \quad \mathrm{B}^* = \frac{\mathrm{B}}{\beta}.$$

The dependencies between the curvature and the pressure are shown in Fig. 31.1. The X-axis is the dimensionless curvature of the deformed tube B^*. The Y-axis is the dimensionless pressure p^*. The solid lines correspond to the Neo–Hookean material ($\nu = 1$). The dashed lines correspond to the Mooney–Rivlin material ($\nu = 0.5$). For small strains ($\mathrm{B}^* > 0.85$) the influence of the material parameter ν on curves $\mathrm{B}^* - p^*$ is very small. The initial curvature β has a small influence to the curves $\mathrm{B}^* - p^*$.

It is found that there is a maximum pressure for each considered material. The maximum pressures p^*_{max} and the corresponding curvatures are presented in Table 31.1. The limiting case of the inflation of a straight tube has a maximum pressure. It is presented for $\beta = 0$ in Table 31.1. The value of p^*_{max} slightly decreases when the initial curvature β increases.

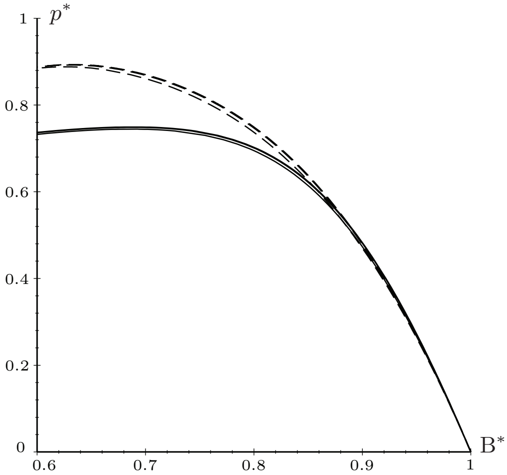

Fig. 31.1 Curvature vs. pressure. The solid lines correspond to the Neo–Hookean material ($\nu = 1$). The dashed lines correspond to the Mooney–Rivlin material ($\nu = 0.5$)

Table 31.1 Maximum pressure

ν	β	B^*	p^*_{max}
	0	–	0.750
	0.05	0.681	0.749
1.0	0.1	0.683	0.749
	0.2	0.692	0.745
	0	–	0.894
	0.05	0.634	0.893
0.5	0.1	0.633	0.892
	0.2	0.630	0.888

The planforms of the deformed curved tubes for the undeformed curvatures $\beta = 0.05$, $\beta = 0.1$ and $\beta = 0.2$ are shown in Fig. 31.2. The gray dotted lines are the initial configuration. The black solid lines correspond to the Neo–Hookean material ($\nu = 1$). The black dashed lines correspond to the Mooney–Rivlin material ($\nu = 0.5$). The deformed curvatures are $B^* = 0.9$, $B^* = 0.8$ and $B^* = 0.7$. For small strain the influence of the material parameter ν on the shape of a deformed tube is very small.

The cross sections of the deformed curved tubes are shown in Fig. 31.3. The gray dotted line is the initial configuration. The left side corresponds to the Neo–Hookean material ($\nu = 1$). The right side corresponds to the Mooney–Rivlin material ($\nu = 0.5$). The black solid lines correspond to the initial curvature $\beta = 0.05$. The gray dashed lines correspond to $\beta = 0.2$. The influence of the initail curvature to the cross section is insignificant for the Neo–Hookean material. The influence of the initail curvature to the cross section is small for the Mooney–Rivlin material. The cross sections differ for the different materials under the high pressures.

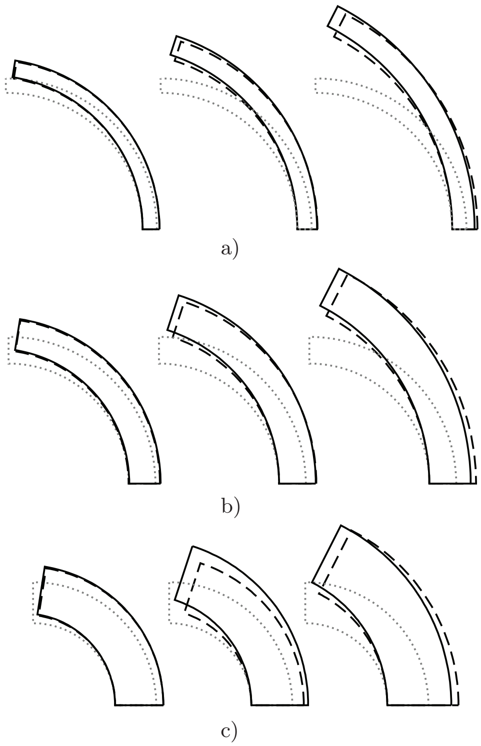

Fig. 31.2 Plane forms of the deformed curved tubes for $B^* = 0.9$, $B^* = 0.8$, $B^* = 0.7$: (a) $\beta = 0.05$; (b) $\beta = 0.1$; (c) $\beta = 0.2$. The gray dotted lines are the initial configuration. The black solid lines correspond to $\nu = 1$. The black dashed lines correspond to $\nu = 0.5$

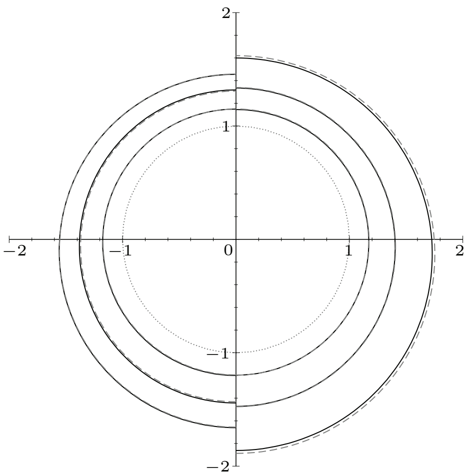

Fig. 31.3 Cross sections of the deformed curved tubes. The gray dotted line is the initial configuration. The left side corresponds to $\nu = 1$. The right side corresponds to $\nu = 0.5$ The black solid lines correspond to $\beta = 0.05$. The gray dashed lines correspond to $\beta = 0.2$

31.5 Conclusions

In this work we study the inflation of membrane which is a sector of torus (curved tube). The problem is considered within the framework of the nonlinear membrane theory. The inflation of curved tube is a special case of pure bending deformation. The governing equations are derived for the tube with arbitrary cross section made of incompressible elastic material. We also consider the inflation of cylindrical membrane (straight tube) to compare with the inflation of curved tube.

The numerical results are presented for the circular cross section and the Mooney-Rivlin material. We obtain that the influence of initial curvature of tube is small to the dependence deformed curvature versus pressure. It is found that there are the maximum pressure for the considered materials. The material parameter ν has a significant influence on the maximum pressure. But for small strains the influence of the material parameter ν is very small. These influence grows when the internal pressure increases and the curvature of the deformed tube decreases. The initial curvature has a small influence on the relation pressure – deformed curvature. If the initial curvature increases then the maximum pressure slightly decreases. It is found that the maximum pressure for straight tube is slightly higher one in curved tube. For lower pressures the shapes of the deformed tubes are closely for the different materials. The difference in the shapes increases with the internal pressure.

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