Ball-Based Shape Processing

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Abstract. Constant radius offsetting and blending operations are important for digital shape and image processing. They may be formulated using Minkowski sums with a ball of fixed radius. We review their extensions to variable distance offsetting. Specifically, we compare three different formulations of variable distance offsetting for planar shapes: orthogonal, radial, and ball. We discuss compatibility conditions that specify when a shape is the offset of another. We also discuss the applications of these formulations for computing the average and morph of two shapes and the centerline of an elongated shape. Finally, we discuss a set theoretic formulation of a variable radius blending of a shape.

Keywords: Variable Radius Offsetting. Rounding and filleting. Medial Axis. Skeleton extraction. Image Segmentation. Shape Averaging. Shape Morphing. Shape correspondence. Tangent Balls. Ball Map. Ball Morph.

1 Introduction

The offset of a shape in the plane plays a central role in shape processing. **Constant distance offsets** (also called constant radius offsets) have been used for modeling safety regions around a shape, for detecting interferences, for simulating growth, for measuring distances and discrepancies between shapes, for defining tolerances on parametric models, and for blending (i.e., removing) sharp corners and narrow passages in the shape or in its complement.

Specializing Serra and Matheron's [1,2] **opening** and **closing** to cases where the structuring element is a ball, Rossignac and Requicha have formulated these **constant-radius blending** operations [3,4] as combinations of **growing** (positive offsetting) and **shrinking** (negative offsetting). Depending on the order, such a combination can either blend all concave or all convex corners, but is not guaranteed to remove both types of sharp features. To address this lack of symmetry and to provide a solutions that blends both concave and convex corners, Williams and Rossignac have defined the **mortar** [5] of the shape as the fill (i.e., grow-shrink) of its boundary and the **tightening** [6,7] of a shape as the result of minimizing the length of the boundary while keeping it inside the mortar.

Variable distance offsets are convenient for expressing one shape as a variation (offset) of another. We discuss here three different formulations of such offsets: orthogonal, radial, and ball.

The **orthogonal offset** [8,9] displaces each point of the boundary of the shape by a prescribed distance in the normal direction.

The **radial offset** adjusts that direction based on the derivative of the distance function and produces a new boundary that is the subset of the envelop [10] of a family of disks with centers on the boundary of the shape and with radii equal to the offset distance value associated with the center point.

The **ball offset** (also called **tangent-ball offset**) recently introduced by Chazal, Lieutier, Rossignac, and Whited [11] produces a new boundary that is the subset of the envelop of a family of disks that are tangent to the original boundary and have as radius half the offset distance associated with the tangent point.

When the distance is constant, all three formulations yield the same result as the constant distance offset. However, when the offset distance varies along the boundary of the shape, each formulation produces a different result and has its advantages.

For example, positive and negative orthogonal offsets of a curve are sometimes used to formulate a variable width stroke in terms of a central line and variable thickness function.

The **medial axis transform** [12,13] formulates a shape as the radial offset of its **medial axis** (curve skeleton). The **pearling** approach to image segmentation [14,15,16], recently introduced by Whited, Rossignac, Slabaugh, Fang, and Unal, computes an approximation of the medial axis transform of a grey-level image in realtime by pushing a disk of variable radius along a path made of desired grey-level pixels.

The **ball morph**, recently introduced by Whited and Rossignac [15,18], expresses one shape as the ball offset of another and uses this formulation to measure and optionally exaggerate local discrepancies between similar shapes or to animate the morph between them.

We say that two shapes are **offset-compatible**, when each one can be expressed as the variable distance offset of the other. The compatibility conditions depend on which formulation of variable offset is used. Chazal, Lieutier, and Rossignac [8,9] have provided sufficient and tight compatibility conditions for the orthogonal offset in terms of the minimum feature size of both sets and of their Hausdorff distance [19]. Subsequently, Chazal, Lieutier, Rossignac, and Whited [11] have shown that the ball offset allows for a similar, but less constraining compatibility condition.

Whited and Rossignac [20] have proposed a set theoretic formulation of variable radius **relative blending** where the blending radius is defined locally relative to a second (control) shape as the radius of the ball that is tangent to both shapes.

In this paper, we review these recent results and explore their interactions. In Section 2, we start with a brief review of the terminology and key concepts that will be used throughout the paper. In Section 3, we discuss constant radius offsetting and its applications to blending and tightening. In Section 4, we present the details of the formulations of the three variable distance offsets and compare their results. In Section 5, we discuss the computation of the offset distance field and mapping between two curves. In Section 6, we discuss offset compatibility conditions. In Section 7, we discuss variable radius relative blending. In Section 8, we discuss the Pearling segmentation of ball-swept structures. In Section 9, we discuss shape morphing and averaging.

2 Review of Key Concepts and Terminology

In this section, we review the basic assumptions and define a few standard, and less standard terms and concepts used throughout the paper.

Topological Domain

For simplicity, we restrict our attention to closed, bounded, manifold, and simply connected planar regions that are hence homeomorphic to a closed disk. We use the letters S and R to denote such regions and the letters P and Q to denote their respective (single loop) bounding curves.

P is a Jordan curve and decomposes the plane into three regions: P , the interior iP of P , and the exterior eP of P . The topologically closed set S is the union of P with iP .

Smoothness and Discrete Models

Some of our theoretical results assume that the bounding curve P of the region of interest S is smooth, and thus has a unique normal outward-pointing direction $N_p(p)$ at each point p of P and that this direction varies continuously along the curve (i.e., it satisfies the Lipschitz condition, see [9,11] for details).

Nevertheless, most of the techniques based on these theoretical results have been successfully applied to shapes bounded by polygonal curves or to discrete models composed of black&white or even grey-level pixels of a regular grid (by using a local estimation of P and of its normal).

Distance and Closest Projection

For simplicity, we will denote by pq the **vector** from point p to point q , which is sometimes written as the “difference” $q-p$ between points.

The **distance** $d(p,q)$ between points p and q may be expressed as the norm $\|pq\|$ of the vector between them.

The **distance** $d(q,P)$ between a point p and a curve P is the minimum of the distances $d(p,q)$ between p and all points q of Q .

The **distance** $d(P,Q)$ between two curves, P and Q , is the minimum of $\|p-q\|$ for all pairs of points p in P and q in Q .

The **closest projection** $c(q,P)$ of point q on curve P is the set of points p such that $d(p,q) = d(q,P)$.

Note that these definitions are not restricted to curves, but also apply to more general sets P and Q .

However, note that when P and Q are smooth curves, if $p = c(q,P)$ and $d(p,q) \neq 0$, then the line segment (p,q) is orthogonal to P at p . Furthermore, if $d(P,Q) \neq 0$, then there exists a point p in P and a point q in Q such that the line segment (p,q) is orthogonal to P at p and to Q at q and that $d(p,q) = d(P,Q)$.

Cut and Medial Axis

Most points q have a single point closest projection $c(q,P)$ on P . The noteworthy exceptions play an important role in shape analysis.

The **cut** $C(P)$ of curve P is the set of points q for which $c(q,P)$ is not a single point. Since the curve P is smooth, it has no common point with $C(P)$.

We differentiate between the interior and the exterior part of the cut. The portion $C_i(P)$ of $C(P)$ in S is called the **interior cut** and is the **medial axis** [12] of S . $C_e(P)$ denotes the **exterior cut** that lies in the **complement** $!S$ of S .

Reach, Regularity, and Minimum Feature Size

A point q in the complement of $C(P)$ may be represented by its closest projection $p = c(q,P)$ on P and of the signed orthogonal offset distance $d(q) = pq \cdot N_P(p)$. The **range**, $[r_i(p), r_e(p)]$, of p is the set of values $d(p)$ for all points q such that $p = c(q,P)$. For example, the range for each point on a circle of radius r is $[-r, \infty]$. The **interior reach** $-r_i(p)$ is the distance $d(p, C_i(P))$ from p to the medial axis of P . the **exterior reach** $r_e(p)$ is the distance $d(p, C_e(P))$ from p to the exterior cut of P .

The **regularity** [5] $r(p)$ of p with respect to P (sometimes also called the **local feature size** or **reach**) is the minimum, $\min(-r_i(p), r_e(p))$, of its two reaches. It is the distance from p to the cut.

Note that, to simplify notation, we assume that these measures are implicitly defined with respect to the curve P . Hence, instead of $r_p(p)$, we simply write $r(p)$.

The **regularity** $r(P)$ of the whole curve P , also called the **minimum feature size** $mfs(P)$ of P , is the minimum of the regularity of its points. Hence, we say that curve P is **r-regular** if $r(P) = r$.

3 Constant Radius Offsetting, Blending, and Tightening

In this section, we discuss operations defined in terms of balls of a fixed radius (a specialization of the standard Minkowski operations) and discuss their applications.

Minkowski Sum, Closing, and Opening

The **Minkowski sum** (also called Minkowski addition or **dilation**) $S+B$ of two sets is the set of points $s+ob$, where point o is a chosen origin, s is a point of S , b a point of B and where ob denotes the vector from point o to point b . Equivalently, $S+B$ may also be defined as the union of sets $B+s$, which are translations of B by vector os , for all points s in S . Hence, $S+B$ may be called the **translational sweep** of the **structuring element** B along S .

The **Minkowski difference** (also called Minkowski subtraction or **erosion**) $S-B$ is defined as the intersection of the sets $S-b$, for all points b in B , which are translations of S by vector bo . Equivalently, it is the set of points x such that B translated by ox lies in S .

Minkowski **closing** of set S by the structuring element B is defined as $(S+B)-B$ and its Minkowski **opening** is defined as $(S-B)+B$.

Minkowski operations are the basic components of **mathematical morphology**. They were developed by Georges Matheron and Jean Serra [1,2].

Constant Radius Offsets

By choosing the structuring element B to be a ball of radius r centered at the origin, Rossignac and Requicha [3] have defined the positive **r-offset** (**grow** or **dilation**) $S^\uparrow$

of set S by a constant distance (also called radius) r as the set of points at distance less or equal to r from S . Equivalently, $S^{\uparrow r}$ is the union of balls with radius r and center in S . Similarly, they define the negative r -offset (**shrinking** or **erosion**) $S^{\downarrow r}$ of S by a constant distance r as the set of points at a distance of more than r from the complement $\bar{S}$ of S . Both are illustrated on Fig. 1.

Fig. 1. The shape S is shown black (left). The material $S^{\uparrow r} - S$ added by dilation is shown grey (left). On the right, the shrunk set $S^{\downarrow r}$ is shown in black and the material $S - S^{\downarrow r}$ removed by erosion is shown in grey.

In their definitions, if S is topologically closed (i.e., contains its boundary), then $S^{\uparrow r}$ and $S^{\downarrow r}$ are also closed.

Hausdorff Distance

The Hausdorff distance, $h(P, Q)$ between two curves (or more generally two sets), P and Q , is the minimum value of r such that $P \subset Q^{\uparrow r}$ and $Q \subset P^{\uparrow r}$. It measures the maximum distance from a point of either set to the other. It is often used to measure the discrepancy between two similar sets, and verifies $h(P, Q) \Leftrightarrow P = Q$.

Filleting and Rounding

Rossignac and Requicha [4] have defined two **constant-radius blending** operations for blending (i.e., rounding) all the concave or all the convex corners of a shape S .

The **filleting** $F_r(S)$ of S (also called its **r-closing** or **r-fill**) is defined as $F_r(S) = (S^{\uparrow r})^{\downarrow r}$. It is the set of points of S that cannot be reached by an open ball B of radius r that is disjoint from S . Hence, $F_r(S)$ has no sharp concave corners and no narrow cracks or holes.

The **rounding** $R_r(S)$ of S , (also called its **r-opening** its **r-round**) is defined as $R_r(S) = (S^{\downarrow r})^{\uparrow r}$. It is the set of points of S that can be reached by a closed ball of radius r in S . Hence, $R_r(S)$ has no sharp convex corners and no narrow protrusions or constrictions.

Note that $R_r(S) \subset S \subset F_r(S)$, but in general $R_r(S) \neq F_r(S)$, see Fig. 2. In fact, as r grows, $R_r(S)$ shrinks and $F_r(S)$ grows.

Fig. 2. On the left, we show the result $R_r(S)$ of rounding in black and the material $S - R_r(S)$ removed by the rounding operation in grey. On the right, we show S in black and the material $F_r(S) - S$ added by filleting in grey.

Williams and Rossignac [5] argue that a set S is **r-regular** if $R_r(S) = F_r(S)$. In other words, S is r-regular if it is equal to its r-closing and to its r-opening.

Some sets cannot be made r-regular by applying any arbitrary combination of r-closing or r-opening, as each application of r-closing leaves sharp convex corners, while the application of r-opening leaves sharp concave corners.

The r-filleting and r-rounding operations defined above are biased. $F_r(S)$ removes concave sharp features and grows the object. $R_r(S)$ removes convex sharp features and shrinks the object.

Mortar and Finite-Scale Topological Operators

Williams and Rossignac [5] define the **r-mortar** $M_r(S)$ of S as the filleting $F_r(P)$ (i.e., the r-closing $(P^\uparrow)^\downarrow_r$) of its boundary P (Fig. 3). In computational geometry, $M_r(S)$ would be called the **alpha hull** of P . Note that $M_r(S) = F_r(S) - R_r(S)$. Note that the r-fill of S is the union $F_r(S) = R_r(S) \cup M_r(S)$ of its rounding with its mortar.

Hence, as shown in Fig. 4, computing the r-mortar “inflates” the boundary of S and decomposes space into three disjoint sets: the **r-interior** $R_r(S)$, the **r-boundary** $M_r(S)$, and the **r-closure** $F_r(S)$. In the limit, as r tends to zero, these operators converge to the standard topological interior, boundary, and closure operators. But for a finite r , they provide an interesting variable-scale version of these topological operators.

Fig. 3. The Mortar (shaded region) of curve P is the space not reachable by an open ball disjoint from P

The r -mortar is one-dimensional (i.e. equal to P) at **r -regular points** p of P , where the regularity $r(p) \geq r$. The r -mortar is a two-dimensional region around the irregular portions of P where $r(p) < r$.

Fig. 4. From left: The set S . The r -rounding $R_r(S)$ in black and the difference $S - R_r(S)$ in grey. S in black and the difference $F_r(S) - S$ in grey. $R_r(S)$ in black and $M_r(S) = F_r(S) - R_r(S)$ in grey.

Stability

Williams and Rossignac [5] define the **stability** of a point q with respect to set S as the smallest value of r for which q belongs to $M_r(S)$. The measure of stability throughout space, not just on the boundary P of S , (see Fig. 5) is a powerful mathematical morphology tool for analyzing how S is imbedded in space. The information it provides cannot be extracted from a topological characterization of S not from a differential analysis of its boundary.

Fig. 5. The stability of a point is proportional to the darkness of the color (capped outside)

Blending Combinations and the Mason Filter

The **blending combinations** $F_r(R_r(S))$ and $R_r(F_r(S))$ tend to eliminate both concave and convex sharp features and often produce r -regular shapes, but in general retain a bias: $F_r(R_r(S))$ is often a strict subset of S and S is often a strict subset of $R_r(F_r(S))$.

One may prefer an unbiased blending operator that is **symmetric** (so that applying it to S or to the complement of S will yield the same result).

Williams and Rossignac [5] show that $R_r(S) \subseteq R_r(F_r(S)) \subseteq F_r(S)$ and $R_r(S) \subseteq F_r(R_r(S)) \subseteq F_r(S)$ and that the application of arbitrary combinations of r -closing and r -opening only alter S in the full dimensional portion of its r -mortar.

Hence, they propose the **Mason filter** [5], which decomposes the two-dimensional portion of the r -mortar into connected components and in each component C of the r -mortar of S , replaces S by either $F_r(R_r(S))$ or $R_r(F_r(S))$: they pick the combination that alters the smallest (in area) portion of C .

Note that Mason acts symmetrically on S and on its complement. Williams and Rossignac show that the output of Mason is guaranteed to have a smaller symmetric difference with the original shape than either $F_r(R_r(S))$ or $R_r(F_r(S))$. Its application to a binary image is shown in Fig. 6.

Fig. 6. The original image (top left) is decomposed (top right) into its r -rounding (black), the complement of its r -filleting (white), and its r -mortar. We compare the result of $F_r(R_r(S))$, shown bottom left, the result of the r -Mason filter (bottom center), and the result of $R_r(F_r(S))$.

As a shape simplification operator, Mason offers advantages over other smoothing or vertex decimation filters.

- 1) It may be used to “regularize” (i.e., remove sharp corners, holes, speckles, thin branches or constrictions) the shape at different scales (Fig. 7).

Fig. 7. The original shape (left) and the results (left-to-right) of applying Mason with increasing values of r

- 2) It is expressed in terms of set-theoretic morphological operators and is hence independent of the representation used for S or P .
- 3) It is self-dual, meaning that $\text{Mason}(S) = \neg(\text{Mason}(\neg S))$, where $\neg X$ denotes the complement of set X , and treats positive and negative space symmetrically.
- 4) It may be viewed as a new tolerance zone that confines the effects of shape simplification to irregular regions, where the boundary has high curvature or where two distinct portions of the boundary are close to each other.
- 5) It preserves the portions of the boundary of S that are regular at the desired scale r .

Tightening

Improving on Mason, Williams and Rossignac propose the **r-tightening** [6,7], which alters S by tightening its boundary in its r -mortar. The r -tightening minimizes the arc-length of P while keeping P inside the r -mortar (Fig. 8). Topological changes may be necessary to produce a new boundary that is smooth (i.e., for which the radius of curvature is never less than r). The computation of the r -tightening is analogous to the computation of the shortest path in a corridor.

Fig. 8. The original shape S (left). Its mortar (grey) and the tightened boundary (center). The tightening of S (right).

Example of tightening in 2D and 3D are shown in Fig. 9 and a comparison with Mason in Fig. 10.

4 Variable Distance Offset Formulations

In this section, we explore offsetting formulations where the offset distance d varies along P . We say that d is the **offset distance field**.

Without loss of generality, we assume that P is oriented (for example, clockwise). With each point p of P , we associate an **offset distance** $d(p)$. The curve obtained by offsetting P by the distance field d will be denoted P^d .

We can also compute the unit **tangent** vector, $t(p)$, and unit outward **normal** vector, $n(p)$, to P at p . For simplicity, we will omit the reference to p in these expression and say that each point p of P is associated with a distance value d , and a **local ortho-normal frame** $\{p, t, n\}$. We will use the notation d' for the **derivative** of the distance function d at p with respect to arc-length parameterization of P .

Fig. 9. Original sets (left) and their tightenings (right)

Fig. 10. Top: Original shape S (left) and its mortar (right). Middle: $Rr(Fr(S))$ and $Fr(Rr(S))$. Bottom: r -Mason (left) and r -tightening (right) of S .

For simplicity, we define the **offset** p^d of p in this local frame as $p^d = p + xt + yn$. This is a convenient formulation for computing the offsets of samples p along P .

We propose three different offsets. We provide here the local constructions of the offset point p^d . Compatibility conditions are discussed further in the paper.

Orthogonal Offset

To differentiate it from other offset formulations, the **orthogonal offset** P^d of curve P by a distance field d is denoted by $O_d(P)$. The orthogonal offset point p^d and the normal vector n^d to P^d at p^d may be computed as:

$$p^d = p + d n \quad \text{and} \quad n^d = (-d' t + n) / \sqrt{(d')^2 + 1}. \quad (1)$$

The value of d does not need to be positive, hence the orthogonal offset curve may be on the left or on the right of P and may cross P when $d=0$.

The line segment (p, p^d) is orthogonal to P at p , but not necessarily to P^d at p^d .

Radial Offset

The **radial offset** P^d of curve P by a distance field d is denoted by $R_d(P)$. The radial offset point and its normal may be computed as:

$$n^d = (-d' t + \sqrt{(1 - d')^2}) n \quad \text{and} \quad p^d = p + d n^d. \quad (2)$$

In portions where d is negative, the radial offset curve will lie on the right of P .

Note that the segment (p, p^d) is orthogonal to P^d at p^d , but not necessarily to P at p .

The radial offset is a portion of the boundary of the region swept by a disk with center p and radius $d(p)$ as p moves along P . As such it is the subset of the envelop [10] of the family of these disks (Fig. 11).

Fig. 11. We show three curves: I (inner), P (central), and O (outer). O is the radial offset by distance field r of P . It is the envelop of a family of circles with centers on P . I is the radial offset of P by the negative $-r$ of the distance field. Note that P is the orthogonal offset by r of I and that O is the ball offset by $2r$ of I .

Ball Offset

The **ball offset** (or tangent ball offset) P^d of curve P by a distance field d is denoted by $B_d(P)$. The ball offset is defined as the composition of the other two offsets: $B_d(P) = R_r(O_r(P))$ with $r=d/2$.

Note that a point p^r of $O_r(P)$ inherits the distance $d/2$ from the corresponding point p on P . However, the derivative r' of that distance with respect to the arc-length parameterization of $O_r(P)$ is in general not equal to $d'/2$. It may be expressed exactly using the curvature of P at p or approximated from a sampling of $O_r(P)$.

The ball offset is a portion of the boundary of the region swept by a disk with radius $d(p)/2$ that is tangent to P at p , as p moves along P . As such, it is the subset of the envelop of the family of these disks.

Comparison and Properties of the Three Offsets

In the special case where $d(p)$ is a constant, $B_d(P) = R_d(P) = O_d(P)$ and hence all three are equal to the constant distance offset discussed earlier. The local disparity between these three offsets increases with the derivative of d .

We compare the three offset formulations in Fig. 12, illustrating their construction at one particular point p of P .

Then P is smooth, the three constructions proposed above establish a homeomorphism between P and P_d . We may use this correspondence to transfer the distance field, as we did above for the construction of the ball offset, even though the derivative of the distance field may not be transferable directly, as offsetting is in general not length preserving. Hence, writing $R_{f(d)}(O_d(P))$, we imply that if a point p of P is associated with an offset distance d , then its image p^d of p by O^d is associated with an offset distance of $f(d)$, where f is some function of d .

Fig. 12. In all three images, we show the same black curve P and indicate its distance field d by the thickness of the shaded region around it (computed as its radial offset). Each curve shows a different offset superimposed on faded versions of the other two for more precise comparison. We show the construction of the offset p^d for a particular point p of P . The normal offset is shown left. The ball offset is shown center. The radial offset is shown right. Note that the radial offset is the furthest from P and that the normal offset is the closest.

We observe that R_d is the inverse of O_d (Fig. 13 left) in the following sense:

$$R_d(O_d(P)) = P \quad \text{and} \quad O_d(R_d(P)) = P \quad (3)$$

It follows that if $O_d(P) = Q$ then $R_d(Q) = P$.

Fig. 13. Left: point q is the orthogonal offset by d of p with respect to P and p is the radial offset by $-d$ of q with respect to P . Hence, $Q=O_d(P)$ and $P=R_d(Q)$. Right: q is the ball offset of p by d with respect to P and p is the ball offset by $-d$ of q with respect to Q . Hence, the ball offset is its own inverse.

Similarly, B_d is its own inverse (Fig. 13 right):

$$B_d(B_d(P)) = P \quad (4)$$

It follows that if $B_d(P) = Q$ then $B_d(Q) = P$.

5 Computation of the Offset Distance Field between Two Curves

We are given two smooth curves, P and Q , and want to express one as the offset of the other. The computation of the distance field depends on which offset is desired.

Orthogonal Map and Offset

For an orthogonal offset [8,9], the distance field d that satisfies $Q = O_d(P)$ may be computed at each point p by constructing a line L that is orthogonal to P at p , and by computing the intersections of L with Q . We distinguish two kinds of intersections: **kissings**, where L osculates Q (i.e., is in tangential contact to Q without crossing it) and **crossings**, where L crosses Q . Note that the number of crossings is always even. Furthermore, the orientation of the crossings alternates along L : as L enters the region R bounded by Q it crosses Q from the left (with respect to the orientation of Q), as it leaves R , it crosses Q from the right.

If for some point p there are no such crossings, then Q cannot be expressed as the orthogonal offset of P .

If there are crossings at each point p , we find the nearest crossing from p in each direction along L . We select the crossing q that has the correct orientation: If L crosses P at p from the left, then L must also cross Q at q from the left. The distance d

associated with p is the dot product $(p-q) \cdot n$, i.e., the signed distance $\|p-q\|$. However, if this segment (p,q) of L contains a kissing or if L is tangential to Q at q , we declare that Q cannot be expressed as the orthogonal offset of P . In these cases, we conclude that P and Q are not O -compatible.

Radial (Closest-Projection) Map and Offset

For a radial offset, we associate with each point p of P the distance $d(p,Q)$ from p to the closest point q on Q . Hence, we map p to its closest projection q on Q .

Ball Map and Offset

For a ball offset [11], we consider the moat X , which is the symmetric difference (XOR), $S \otimes R$ between the region, S and R , bounded respectively by P and Q . Then, we compute the medial axis transform of X .

If X has bifurcations or kinks (points where the normal is discontinuous), we declare that P and Q are not **B-compatible**.

Otherwise, the medial axis defines a family of disks in X that are each tangent to P at exactly one point p and tangent to Q at exactly one point q (Fig. 14). We say that q is the image of p by a **ball map** [11], from P to Q and vice versa. We associate with p and with q the diameter of that tangent disk, but with opposite signs. The sign associated with p is positive when p lies inside R .

The distance field d assigned to P satisfies $B_d(P) = Q$. We also have $B_{-d}(Q) = P$.

Fig. 14. When the maximal disks in the moat between the two curves P and Q touch each curve at a single point, they define the correspondence (ball-map) between a point p of P and a point q of Q . Furthermore, the diameter of the touching ball is the ball offset distance associated with these contact points.

6 Compatibility

We say that a distance field d is **X-compatible** with a curve P , where ‘ X ’ stands for ‘ O ’, ‘ B ’, or ‘ R ’, when the X -offset of P by d is free from cusps and self-intersections.

Incompatibilities may be produced **locally**, when for example the orthogonal offset distance exceeds the local radius of curvature (this is sometimes called the fish-tail and illustrated in an inward offset of an ellipse) or when the derivative of the distance field of a radial offset exceeds 1 (the disk of radius d at p is contained inside the disk of radius $d+\epsilon d'$ centered at a nearby point $p+\epsilon t$, for an infinitely small ϵ).

Incompatibilities may also be produced **globally**, where the images on P^d of two disjoint parts of P intersect.

We say that two curves P and Q are **X-compatible** if each one can be expressed as the X-offset of the other (i.e. if a suitable X-compatible distance field may be found).

Note that, because an orthogonal offset is the inverse of a radial offset, O-compatibility implies B-compatibility and vice versa.

For example, Fig. 15 shows two curves that are B-compatible, but are not O-compatible, and hence not R-compatible either.

Chazal, Lieutier, and Rossignac [8,9] prove that two smooth curves P and Q are O-compatible (and hence also R-compatible) if

$$h(P,Q) < (2 - \sqrt{2}) \min(r(P), r(Q)) \quad (5)$$

This sufficient, although not necessary condition bounds the disparity (measured in terms of Hausdorff distance) as a function of the minimum feature sizes (or regularity) of the two curves. They also show that the constant $(2 - \sqrt{2})$ is tight by producing an example of two curves that are not O-compatible (because they intersect at right angles) and for which we have $h(P,Q) = (2 - \sqrt{2}) \min(r(P), r(Q))$.

Chazal et al. [11] prove that two smooth curves P and Q are B-compatible if

$$h(P,Q) < \min(r(P), r(Q)) \quad (6)$$

This bound is also tight, but is less constraining, since it allows a greater disparity between the curves for the same feature size. Hence, we conclude that B-compatibility is easier to achieve than O-compatibility.

Fig. 15. The two curves are B-compatible (i.e., each one is the ball offset of the other). The correspondence (ball map) is shown on the left as line segments joining points p to corresponding points q . But the two curves are not O-compatible, nor R-compatible, as shown on the right.

7 Variable Radius Relative Blending

When blending (i.e., removing sharp features of) a shape S , one may want the radius of the blending ball to vary along the boundary P of the shape S . This objective raises two challenges:

- 1) How to define the radius function
- 2) How to define the result of the blend

Whited and Rossignac [20] propose their variable-radius **relative blending** approach where the radius field used to blend S is defined as half the ball offset distance from the boundary P of S to a control curve Q .

Their approach removes the portions of P that are incompatible with Q and replaces them with smoothly connecting, circular-arc blends, as shown in Fig. 16. This operation corresponds to the removal of dangling branches of the medial axis. The result is not truly compatible with the control curve, since the maximal ball centered at the bifurcation of the original medial axis has more than one contact point with it, but is pseudo-compatible and hence, one can establish a ball map between the two curves and can express the new rounded curve as the ball offset of the control curve.

Fig. 16. The outer-curve (left) has a protrusion (top) that is incompatible with the inner (mouth-like) control curve. This incompatibility is indicated by a bifurcation of the medial-axis of the symmetric difference between the two curves. On the right, the protrusion has been removed (dotted lines) and was replaced by a circular arc.

Furthermore, Whited and Rossignac [20] provide a set-theoretic formulation of the desired result in terms of S , the union C of the maximal balls in the symmetric difference between P and R that touch both P and Q , and the set M bounded by the trimmed medial axis loop (completed by adding the intersections and overlap between P and Q). The relative blending $B_R(S)$ of set S with respect to set R is

$$B_R(S) = (S \cap (M \cup C)) \cup (M - C) \quad (7)$$

They show that this approach may be used in three dimensions to simultaneously blend the concave and convex features of a 3D shape with variable radius fillets and rounds (Fig. 17).

Fig. 17. The semitransparent plane (left) is used as a control surface to blend a complex corner of a polygonal solid. The resulting shape (right) contains variable-radius blends.

8 Pearling Segmentation

Whited et al. [14,15] have proposed an interactive technique (called **Pearling**) for segmenting strokes in images and also tubular structures in 3D medical data sets. For example, to trace a road or artery in an image (Fig. 18), they start from a user-provided position and direction on the road and create, one at a time, a string of balls along the road. The position and radius of each ball are adjusted using a few iterations so that the ball is not far from the previous one and so that its center is mostly filled with “good” pixels that have “road-like” colors and that a small periphery on opposite

Fig. 18. A particular street is traced by Pearling in real-time. Upon request, the tracing may expand to branching streets.

sides is filled with “bad” pixels that repel the center of the ball. To define “good” and “bad” pixels, the user selects sample good and bad regions by painting over a portion of the road and one or more portions of its surrounding.

The construction is interactive (the road is traced as fast as a rough centerline can be drawn by the user) and can track bifurcation. A smooth curve that interpolates the centers of the balls may be viewed as an approximation of the **centerline** of the road. Treating it as the medial axis leads to the formulation of the segmented region in terms of its medial axis transform. Hence, away from the bifurcation points, the left and right borders of the road are the radial offsets of the centerline and one border is the ball-offset of the other. This observation is important for subsequent processing of the results, for example to remove the bulges associated with possible, but not traced bifurcations.

In 3D, Pearling has also been used [16] to trace human vasculature (Fig. 19) from discrete medical scans. A similar approach was developed for tracing tubes in solids represented by triangle meshes [21,22].

Fig. 19. Tubular structures in human anatomies segmented by Pearling from medical scans

9 Shape Morphing and Midarc Averaging

In their **ball morph** approach to in-betweening, Whited and Rossignac [17,18] propose to use the ball map correspondence [11] to define a curved trajectory from each point of curve P to a single point of a B -compatible curve Q . The trajectory is a circular arc that is orthogonal to p at P and to q at Q (Fig. 20 left).

They establish these ball map correspondences and trajectories for a coordinated sampling of P and Q , sample the trajectories uniformly, and finally join each set of corresponding samples by a closed-loop curve (Fig. 20 right). These curves may be used as **frames** of an in-betweening animation (Fig. 21). More importantly, this formulation also defines a **continuous animation model**.

Fig. 20. The circular trajectory from p to q is fully defined by the corresponding maximal disk

The halfway frame (which joins the mid-course point on each arc) provides a new definition of the **average** of two shapes. We call it the **midarc average**. It differs from the medial axis of the gap between the two curves (Fig. 22 left).

This formulation may be extended to produce a new type of centerline (with bifurcations) of a single shape. We call it the **midarc axis**. We contrast it (Fig. 22 right) with prior definitions: Blum's Medial Axis [MAT Blum], Layton's PISA [23], and Asada and Brady's Smooth Locus of Symmetry [24].

Fig. 21. Trajectories (top left) and frames of a Ball morph animation between an apple and a pear

Fig. 22. The midarc average of two curves is the mid-arc axis of the gap between them. It is superimposed (left) over the darker medial axis. The construction of the midarc axis is contrasted (right) with other definitions of centerlines.

10 Conclusion

We have briefly reviewed a variety of ball-based techniques for analyzing, transforming, comparing, and morphing shapes and have contrasted them with closest-projection techniques. Specifically, ball-based techniques may be used to blend shapes so that they are B-compatible and to create a correspondence between B-compatible shapes which can be used to model one shape as the ball offset of another, to compute weighted averages of shapes, and to produce in-betweening animations that smoothly morph from one shape to another.

Several challenges remain.

One challenge is the closure of the domain. In two dimensions, these techniques have been implemented for discrete models with regularly spaced samples (pixels) and with ordered samples spaced along the curve, for polygonal models, and for piecewise-circular (PCC) [25] representations of the curves. The first two do work in practice when the sampling density is sufficient, but are theoretically incorrect. The latter is theoretically correct and efficient, but may produce derived offset curves (that are not PCC and hence must be approximated by a PCC if one wishes to offset them again).

Another challenge is the extension of these techniques to three dimensions. Using efficient (hardware-assisted) algorithms for computing offsets and distance fields permits to implement most of the techniques discussed here on discrete (voxel) and point cloud representations [26] and also on triangle-meshes [11]. However, the formulation of the tightening of a three-dimensional shape is not as straightforward, because the minimization of the surface area within the 3D mortar does not in general yield an acceptable solution. This theoretical extension is the topic of the forthcoming PhD dissertation of student Jason Williams.

A third challenge is the evaluation of the practical and perceptual benefits of using ball offsets rather than orthogonal or radial offsets for drawing variable thickness curves and of using the midarc axis rather than the medial axis as a skeletal abstraction of a shape.

A fourth challenge is the extension of these approaches to more than two curves or more than two surfaces. This aspect is currently explored by the author in collaboration with Drs. Raimund Seidel and Brian Wyvill.

Finally, one may wish to investigate further the relation between the variable radius relative blending and the variable distance offsets discussed here and the adaptive neighborhood operations used in mathematical morphology [27]. Similarly, one may wish to investigate the further relation between the Mason and Tightening filters discussed above and the Alternating Sequential Filters used in mathematical morphology [28].

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References

1. Serra, J.: *Image Analysis and Mathematical Morphology*. Academic Press, New York (1982)
2. Matheron, G.: *Random Sets and Integral Geometry*. J Wiley Sons, New York
3. Rossignac, J., Requicha, A.: Offsetting Operations in Solid Modelling. *Computer-Aided Geometric Design* 3, 129–148 (1986)
4. Rossignac, J., Requicha, A.: Constant-Radius Blending in Solid Modeling. In: *ASME Computers In Mechanical Engineering (CIME)*, vol. 3, pp. 65–73 (1984)
5. Williams, J., Rossignac, J.: Mason: Morphological Simplification. *Graphical Models* 67(4), 285–303 (2005)
6. Williams, J., Rossignac, J.: Tightening: Curvature Limiting Morphological Simplification. In: *ACM Symposium on Solid and Physical Modeling (Sketch)*, pp. 107–112. MIT, Cambridge (June 2005)
7. Williams, J., Rossignac, J.: Tightening: Morphological Simplification. *International Journal of Computational Geometry & Applications (IJCGA)* 17(5), 487–503 (2007)
8. Chazal, F., Lieutier, A., Rossignac, J.: Orthomap: Homeomorphism-guaranteeing normal projection map between surfaces. In: *ACM Symposium on Solid and Physical Modeling*, pp. 9–14. MIT, Cambridge (June 2005)
9. Chazal, F., Lieutier, A., Rossignac, J.: Normal map between normal-compatible manifolds. *International Journal of Computational Geometry & Applications (IJCGA)* 17(5), 403–421 (2007)
10. Monge, G.: *Applications de l'analyse a la geometrie*, 5th edn., Bachelier, Paris (1894)
11. Chazal, F., Lieutier, A., Rossignac, J., Whited, B.: Ball Map: Homeomorphism between compatible surfaces. *International Journal of Computational Geometry and Applications (CG&A)* 20(3), 285–306 (2010)
12. Blum, H.: A Transformation for Extracting New Descriptors of Shape. In: Wathen-Dunn, W. (ed.) *Models for the Perception of Speech and Visual Form*, pp. 362–380. MIT Press, Cambridge (1967)

13. Foskey, M., Lin, M., Manocha, D.: Efficient computation of a simplified medial axis. In: ACM Symposium on Solid Modeling and Applications, pp. 96–107 (2003)
14. Whited, B., Rossignac, J., Slabaugh, G., Fang, T., Unal, G.: Pearling: Stroke segmentation with crusted pearl strings. In: The First International Workshop on Image Mining Theory and Applications, IMTA (2008)
15. Whited, B., Rossignac, J., Slabaugh, G., Fang, T., Unal, G.: Pearling: Stroke segmentation with crusted pearl strings. *Journal of Pattern Recognition and Image Analysis (PRIA)* 19(2), 277–283 (2009)
16. Whited, B., Rossignac, J., Slabaugh, G., Fang, T., Unal, G.: Pearling: 3D interactive extraction of tubular structures from volumetric images. In: MICCAI Workshop: Interaction in Medical Image Analysis and Visualization (2007)
17. Whited, B., Rossignac, J.: B-morphs between b-compatible curves. In: ACM Symposium on Solid and Physical Modeling, SPM (2009)
18. Whited, B., Rossignac, J.: Ball morph: Definition, Implementation, and Comparative Evaluation. *IEEE Trans. on Visualization & Computer Graphics* (2011), <http://www.computer.org/portal/web/csdl/doi/10.1109/TVCG.2010.115>
19. Atallah, M.: A linear time algorithm for the Hausdorff distance between convex polygons. *Information Processing Letters* 16, 207–209 (1983)
20. Whited, B., Rossignac, J.: Relative blending. *Journal of Computer-Aided Design (JCAD)* 41(6), 456–462 (2009)
21. Mortara, M., Patane, G., Spagnuolo, M., Falcidieno, B., Rossignac, J.: Plumber: A method for a multi-scale decomposition of 3D shapes into tubular primitives and bodies. In: ACM Symposium on Solid Modeling (2004)
22. Mortara, M., Patane, G., Spagnuolo, M., Falcidieno, B., Rossignac, J.: Blowing Bubbles for the Multiscale Analysis and Decomposition of Triangle-Meshes. *Algorithmica* 38(1) (2004)
23. Leyton, M.: *Symmetry, Causality, Mind*. MIT Press, Cambridge (1992)
24. Asada, H., Brady, M.: The curvature primal sketch. *IEEE Trans. Pattern Anal. Mach. Intell.* 8(1), 2–14 (1986)
25. Rossignac, J., Requicha, A.: Piecewise-Circular Curves for Geometric Modeling. *IBM Journal of Research and Development* 13, 296–313 (1987)
26. Chen, Y., Wang, H., Rosen, D., Rossignac, J.: Filleting and rounding using a point-based method. *ASME Design Engineering Technical Conferences, DETC05/DAC-85408* (September 2005)
27. Debayle, J., Pinoli, J.C.: Adaptive neighborhood mathematical morphology and its applications to image filtering and segmentation. In: *European Congress for Image Analysis and Stereology* (2005)
28. Morales, A., Acharya, R.: Alternating sequential filters and multiresolution morphology. In: *IEEE International Conference on Systems Engineering*, pp. 534–538 (1990), doi:10.1109/ICSYSE.1990.203212