On the Size of Graphs That Admit Polyline Drawings with Few Bends and Crossing Angles

Eyal Ackerman¹, Radosław Fulek², and Csaba D. Tóth^{3,*}

¹ University of Haifa at Oranim
ackerman@sci.haifa.ac.il

² Ecole Polytechnique Fédérale de Lausanne
radoslaw.fulek@epfl.ch

³ University of Calgary
cdtoth@math.ucalgary.ca

Abstract. We consider graphs that admit polyline drawings where all crossings occur at the same angle $\alpha \in (0, \frac{\pi}{2}]$. We prove that every graph on n vertices that admits such a polyline drawing with at most two bends per edge has $O(n)$ edges. This result remains true when each crossing occurs at an angle from a small set of angles. We also provide several extensions that might be of independent interest.

1 Introduction

Graphs that admit polyline drawings with few bends per edge and such that every crossing occurs at a large angle have received some attention lately, since cognitive experiments [7,8] indicate that such drawings are almost as readable as planar drawings. That is, even though they may contain crossings, one can still easily track the edges of such drawings.

A *topological graph* is a graph drawn in the plane with vertices represented by distinct points and edges as arcs connecting its vertices, but not passing through any other vertex. A *polyline drawing* of a graph G is a topological graph where each edge is drawn as a simple polygonal arc between the incident vertices, but not passing through any bend point of other arcs. In a polyline drawing, every crossing occurs in the relative interior of two segments of the two polygonal arcs, and so they have a well-defined crossing angle in $(0, \frac{\pi}{2}]$.

Didimo *et al.* [5] introduced *right angle crossing (RAC)* drawings, which are polyline drawing where all crossings occur at a right angle. They prove that a graph with $n \geq 3$ vertices that admits a straight line RAC drawing has at most $4n - 10$ edges, and this bound is best possible. A simpler proof of this bound was later found by Dujmović *et al.* [6]. It is not hard to show that any graph admits an RAC drawing with three bends per edge (see Figure 1 for an example). Arikushi *et al.* [3] have recently proved, improving previous results by Didimo *et al.* [5], that if a graph with n vertices admits an RAC drawing with at most two bends per edge, then it has $O(n)$ edges.

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Fig. 1. An RAC drawing of K_6 with 3 bends per edge

Dujmović *et al.* [6] generalized RAC drawings, allowing crossings at a range of angles rather than at right angle. They considered αAC drawings, which are polyline drawings where every crossing occurs at angle *at least* α . They showed that a graph with n vertices and a straight line αAC drawing has at most $\frac{\pi}{\alpha}(3n - 6)$ edges, by partitioning the graph into $\frac{\pi}{\alpha}$ planar graphs. They also proved that their bounds are essentially optimal for $\alpha = \frac{\pi}{k} - \varepsilon$, with $k = 2, 3, 4, 6$ and sufficiently small $\varepsilon > 0$.

Results for polyline drawings. We consider polyline drawings where every crossing occurs at the same angle $\alpha \in (0, \frac{\pi}{2}]$. An αAC_b^- drawing of a graph is a polyline drawing where every edge is a polygonal arc with at most b bends and every crossing occurs at angle *exactly* α . It is easy to see that every graph with n vertices that admits an αAC_0^- drawing has at most $3(3n - 6)$ edges (see Lemma 1 below). Every graph admits an αAC_3^- drawing for every $\alpha \in (0, \frac{\pi}{2}]$: Didimo *et al.* [5] constructed an RAC drawing of the complete graph with three bends per edge (see also Figure 1), where every crossing occurs between a pair of orthogonal segments of the same orientation, so an affine transformation deforms all crossing angles uniformly. It remained to consider graphs that admit αAC_1^- or αAC_2^- drawings. Our main result is:

Theorem 1. *For every $\alpha \in (0, \frac{\pi}{2}]$, a graph on n vertices that admits an αAC_2^- drawing has $O(n)$ edges.*

For $\alpha = \frac{\pi}{2}$, this has recently been proved by Arikushi *et al.* [3]. Their proof techniques, however, do not generalize to all $\alpha \in (0, \frac{\pi}{2}]$. Our techniques use some ideas from [3], but are in fact somewhat simpler. The constant hidden in the big- O notation in Theorem 1 is quite moderate. For graphs admitting αAC_1^- drawings (with at most one bend per edge), we give a better upper bound than Theorem 1, using a simpler proof.

Theorem 2. *For any angle $\alpha \in (0, \frac{\pi}{2}]$, a graph on n vertices that admits an αAC_1^- drawing has at most $27n$ edges.*

A straightforward generalization of αAC_2^- drawings are polyline drawings where each crossing occurs at an angle from a list of k distinct angles.

Theorem 3. *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. If a graph G on n vertices admits a drawing with at most two bends per edge such that every crossing occurs at some angle from A , then G has $O(k^3n)$ edges.*

Generalizations to topological graphs. Suppose that every edge in a topological graph is partitioned into edge-segments, such that all crossings occur in the relative interior of the segments. The bends in polyline drawings, for example, naturally define such edge partitions. An *end segment* is an edge-segment incident to a vertex of the edge, while a *middle segment* is an edge-segment not incident to any vertex. The key idea in proving Theorems 1 and 2 is to consider the crossings that involve either two end segments, or an end segment and a middle segment. This idea extends to topological graphs whose edge segments satisfy a few properties (which automatically hold for polyline drawings with same angle crossings). We obtain the following results, which might be of independent interest.

Theorem 4. *Let $G = (V, E)$ be a topological graph on n vertices, in which every edge can be partitioned into two end segments, one colored red and the other colored blue, such that:*

- (1) *no two end segments of the same color cross;*
- (2) *every pair of end segments intersects at most once; and*
- (3) *no end segment is crossed by more than k end segments that share a vertex.*

Then G has $O(kn)$ edges.

Theorem 5. *Let $G = (V, E)$ be a topological graph on n vertices. Suppose that every edge of G can be partitioned into two end segments and one middle segment such that:*

- (1) *each crossing involves one end segment and one middle segment;*
- (2) *each middle segment and end segment intersect at most once; and*
- (3) *each middle segment crosses at most k end segments that share a vertex.*

Then G has $O(k^2n)$ edges.

Organization. We begin with a few preliminary observations. Our main results appear in Section 3 where we prove Theorems 1, 2, 3, and 5. Theorem 4 is proved in Section 4. It is actually not needed for the proof of Theorem 2, however, it also implies that graphs with αAC_1^- drawings have a linear number of edges.

2 Preliminaries

In a polyline drawing of a graph, the edges are simple polygonal paths, consisting of line segments. We start with a few initial observations about line segments and polygonal paths. We say that two line segments *cross* if their relative interiors intersect in a single point. (Note that intersecting segments that share an endpoint or are collinear do not cross.)

Lemma 1. *Let $\alpha \in (0, \frac{\pi}{2}]$, and let S be a finite set of line segments in the plane such that any two segments may cross only at angle α . Then S can be partitioned into at most three subsets of pairwise noncrossing segments. Moreover, if $\frac{\pi}{\alpha}$ is irrational or if $\frac{\pi}{\alpha} = \frac{p}{q}$, where $\frac{p}{q}$ is irreducible and q is even, then S can be partitioned into at most two subsets of pairwise noncrossing segments.*

Proof. Partition S into maximal subsets of pairwise parallel line segments. Let $\mathcal{S}$ denote the subsets of S . We define a graph $G_{\mathcal{S}} = (\mathcal{S}, E_{\mathcal{S}})$, in which two subsets $S_1, S_2 \in \mathcal{S}$ are joined by an edge if and only if their respective directions differ by angle α . Clearly, the maximum degree of a vertex in $G_{\mathcal{S}}$ is two. Hence, $G_{\mathcal{S}}$ is 3-colorable. In any proper 3-coloring of $G_{\mathcal{S}}$, the union of each color class is a set of pairwise noncrossing segments in S , since they do not meet at angle α .

If $\frac{\pi}{\alpha}$ is irrational, then $G_{\mathcal{S}}$ is cycle-free. If $\frac{\pi}{\alpha} = \frac{p}{q}$, where $\frac{p}{q}$ is irreducible and q is even, then $G_{\mathcal{S}}$ can only have even cycles. In both cases, $G_{\mathcal{S}}$ is 2-colorable, and S has a partition into two subsets of pairwise noncrossing segments.

Fig. 2. (a) The turning angles of a polygonal path. (b) Two crossing polygonal paths with the same turning angle between p and q . (c) Two noncrossing polygonal paths with the same turning angle between p and q .

Consider a simple open polygonal path $\gamma = (v_0, v_1, \dots, v_n)$ in the plane. At every interior vertex v_i , $1 \leq i \leq n-1$, the *turning angle* $\angle(\gamma, v_i)$ is the directed angle in $(-\pi, \pi)$ (the counterclockwise direction is positive) from ray $\overrightarrow{v_{i-1}v_i}$ to $\overrightarrow{v_iv_{i+1}}$; see Figure 2(a). The *turning angle* of the polygonal path γ is the sum of turning angles $\sum_{i=1}^{n-1} \angle(\gamma, v_i)$. We say that two line segments *overlap* if one of them is contained in the other.

Lemma 2. *Let p and q be two points in the plane. Let γ_1 and γ_2 be two directed simple polygonal paths from p to q . If γ_1 and γ_2 have the same turning angle and they do not cross, then the first segment of γ_1 overlaps with the first segment of γ_2 and the last segment of γ_1 overlaps with the last segment of γ_2 .*

Proof. Let $\gamma_1 = (u_0, u_1, \dots, u_m)$ and $\gamma_2 = (v_0, v_1, \dots, v_n)$, with $p = u_0 = v_0$ and $q = u_m = v_n$. Let β be their common turning angle. Since γ_1 and γ_2 do not cross, they enclose a weakly simple polygon P with $m+n$ vertices (Figure 2(c)). Suppose w.l.o.g. that the vertices of P in clockwise order are $v_0 = u_0, u_1, \dots, u_m = v_n, v_{n-1}, \dots, v_1$. Every interior angle of P is in $[0, 2\pi]$, and the sum of interior angles is $(m+n-2)\pi$. The sum of interior angles at

the vertices $u_1, \dots, u_{m-1}$ is $(m-1) \cdot \pi + \beta$; and the sum of interior angles at $v_1, \dots, v_{n-1}$ is $(n-1) \cdot \pi - \beta$. Hence the interior angles at p and q are both 0.

Let G be a topological multi-graph. We say that two edges *overlap* if their intersection contains a connected set of more than one point. Such a maximal set is called an *overlap* of the two edges. A *common tail* is an overlap of two edges that contains a vertex adjacent to both edges.

Lemma 3. *Let G be a topological multi-graph in which some edges may overlap, but only in common tails. Then the edges of G can be slightly perturbed such that all overlaps are removed and no new crossings are introduced.*

Proof. We successively perturb G and decrease the number of edge pairs that have a common tail. Let $e = (u, v)$ be an edge in G , and let $e_1, e_2, \dots, e_k$ be edges in G that have a common tail with e , such that their overlaps with e contain the vertex u . Direct all these edges away from u . Then every edge e_i follows an initial portion of e , and then turns either right or left at some *turning point* p_i . Suppose that there is at least one right turning point, and let p_j be the last such point. (Observe that the part of e between u and p_j does not contain turning points of common tails adjacent to vertex v , for otherwise there would be overlaps that are not common tails.)

Fig. 3. Removing overlaps

Redraw all the edges e_i with a right turning point such that they closely follow e on the right. See Figure 3. We have removed the overlap between e and e_j , and decreased the number of edge pairs that have a common tail. If there are no right turning points, then we redraw all the edges with a left turning point such that they closely follow e on the left.

We say that a topological graph is *simple* if any two of its edges meet at most once, either at a common endpoint or at a crossing. By a *quasi-planar* graph we understand a topological graph with no three pairwise crossing edges. In the sequel we will use the linear upper bound on the maximum number of edges in a simple quasi-planar graph by Ackerman and Tardos [1].

3 Polyline Drawing with the Same Crossing Angle

In this section, we prove Theorems 1 and 2. The key new technique can be summarized as follows. Fix an αAC_2^∞ drawing of a graph $G = (V, E)$. For a constant fraction of the edges $(u, v) \in E$, we draw a new “red” edge that connects u to another vertex in V (which is not necessarily v). The red edges closely follow edges in the αAC_2^∞ drawing of G , and they only turn at bends and some crossings of G . We show that all crossings among red edges occur on their end segments. This combined with Lemma 1 will complete the proof. We continue with the details.

An αAC_∞^∞ drawing of a graph G is polyline drawing where every crossing occurs at angle α . Every edge is a polygonal arc that consists of line segments. The first and last segments of each edge are called *end segments*, all other segments are called *middle segments*. Note that each end segment is incident to a vertex of G . Let $G = (V, E)$ be a graph with an αAC_∞^∞ drawing. It is clear that G has at most $3n - 6$ crossing-free edges, since they form a plane graph. All other edges have some crossings. We distinguish several cases below depending on whether the edges have crossings along their end segments.

3.1 Crossings between End Segments

Lemma 4. *Let $0 < \alpha \leq \frac{\pi}{2}$. Let $G = (V, E)$ be a graph on $n \geq 4$ vertices that admits an αAC_∞^∞ drawing such that an end segment of every edge $e \in E$ crosses an end segment of some other edge in E . Then $|E| \leq 36n$. Moreover, the number of edges in E whose both end segments cross some end segments is at most $18n$.*

Proof. In the αAC_∞^∞ drawing of G , let S be the set of end segments that cross some other end segments. We have $|E| \leq |S| \leq 2|E|$. Direct each segment $s \in S$ from an incident vertex in V to the other endpoint. For a straight line edge, choose the direction arbitrarily.

We construct a directed multi-graph $G' = (V, R)$. We call the edges in R *red*, to distinguish them from the edges of E . For every end segment $s \in S$, we construct a red edge $\gamma(s)$, which is a polygonal path with one bend between two vertices in V . For a segment $s \in S$, the path $\gamma(s)$ is constructed as follows.

Let $u_s \in V$ denote the starting point of s (along its direction). Let c_s be the first crossing of s with an end segment, which we denote by t_s . Let $v_s \in V$ be a vertex incident to the end segment t_s . Now let $\gamma(s) = (u_s, c_s, v_s)$.

Note that for every $s \in S$, the first segment of $\gamma(s)$ is part of the segment s and does not cross any segment in S . Hence the first segments of the red edges $\gamma(s)$ are distinct and do not cross other red edges. However, the second segment of $\gamma(s)$ may cross other red edges or overlap with other red edges. Since the edges of G cross at angle α and c_s is a crossing, the turning angle of $\gamma(s)$ is $\pm\alpha$ or $\pm(\pi - \alpha)$. Note that overlaps may occur only between red end segments. However, they can be removed using Lemma 3.

We show that for any two vertices $u, v \in V$, there are at most 4 directed red edges from u to v . By Lemma 2, any two noncrossing paths of the same turning

angle between u and v must overlap in the first and last segments. The first segments of the red edges are pairwise non-overlapping. Furthermore, the red edges from u to v cannot cross, since the segments incident to v cannot cross. Since red edges may have up to 4 distinct turning angles, there are at most 4 pairwise noncrossing edges between u and v . Hence there are at most 8 red edges between u and v (in either direction).

Distinguish two types of red edges. Let R_1 be the set of red edges whose second segment crosses some other red edge, and let R_2 be the set of red edges where both segments are crossing-free. Note that two edges in R_1 cannot follow the same path γ in opposite directions. If R_2 contains two edges that follow the same path γ in opposite directions, then remove one arbitrarily, and denote the remaining edges by $R'_2 \subseteq R_2$, with $|R'_2| \geq \frac{1}{2}|R_2|$.

Hence, there are at most 4 red edges in R_1 between any two vertices in V . Let S_1 be the set of second segments of the red edges in R_1 . By Lemma 1, there is a subset $S'_1 \subseteq S_1$ of pairwise noncrossing segments of size at least $\frac{1}{3}|R_1|$. Let R'_1 be the set of red edges containing the segments S'_1 , with $|R'_1| \geq \frac{1}{3}|R_1|$. Now $(V, R'_1 \cup R'_2)$ is a simple multi-graph with maximum multiplicity 4, with at most $4(3n - 6)$ edges. Hence, $|R| \leq 3 \cdot 4(3n - 6) = 36n - 72$, if $n \geq 3$.

For the last part of the statement observe that in the above argument, an edge in E is counted twice if its both end segments are in S .

Theorem 2 follows from Lemma 4.

Proof of Theorem 2: Let $G = (V, E)$ be a graph with $n \geq 4$ vertices drawn in the plane with an αAC_1^- drawing. Let $E_1 \subseteq E$ denote the set of edges in E that have at least one crossing-free end segment. Let $G_1 = (V, E_1)$ and $G_2 = (V, E \setminus E_1)$.

It is easy to see that if $\alpha \neq \frac{\pi}{3}$, then G_1 is a simple quasi-planar graph and so it has at most $6.5n - 20$ edges by a result of Ackerman and Tardos [1]. If $\alpha = \frac{\pi}{3}$, let S_1 be the set of crossed end segments of edges in E_1 . By Lemma 1, there is a subset $S'_1 \subseteq S_1$ of pairwise noncrossing segments of size $\frac{1}{3}|E_1|$. The graph G'_1 corresponding to these edges is planar, with at most $3n - 6$ edges. Hence E_1 contains at most $3 \cdot (3n - 6) = 9n - 18$ edges.

By Lemma 4, G_2 has at most $18n$ edges. Hence, G has at most $24.5n$ edges if $\alpha \neq \frac{\pi}{3}$ and at most $27n$ edges otherwise. $\square$

Remark. It is easy to generalize the proof of Lemma 4 to the case that every two polyline edges cross at one of k possible angles. The only difference is that the red edges may have up to $2k$ different turning angles.

Lemma 5. *Let $G = (V, E)$ be a graph on $n \geq 4$ vertices that admits a polyline drawing such that an end segment of every edge $e \in E$ crosses an end segment of some other edge in E at one of k possible angles. Then $|E| \leq 36kn$.* $\square$

Corollary 1. *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. If a graph G on n vertices admits a drawing with at most one bend per edge such that every crossing occurs at some angle from A , then G has at most $(36k + 3)n$ edges.* $\square$

3.2 Crossings between End Segments and Middle Segments

We prove Theorem 5. Note that in an αAC_2^- drawing, we can take $k = 2$.

Proof of Theorem 5: Without loss of generality we can assume that G is drawn in the plane so that the number of edge crossings is minimized under the constraint that it satisfies conditions (1)–(3). Observe that G has at most $3n - 6$ edges whose end segments are crossing-free, since two such edges cannot cross each other. Let $E_1 \subset E$ denote the set of edges with at least one crossed end segment. Let S_1 be the set of crossed end segments. It is clear that $|E_1| \leq |S_1| \leq 2|E_1|$. We will construct a red edge for at least $\frac{1}{k}|S_1|$ end segments in S_1 .

Direct every end segment from the incident vertex in V to its other endpoint, and direct every middle segment arbitrarily. For every end segment $s \in S_1$, let c_s be the first crossing along s (that is, closest to its incident vertex in V).

By property (3), for every middle segment m and for every vertex $v \in V$, there are at most k end segments s adjacent to v with $c_s \in m$. Order these at most k end segments according to the position of their crossings c_s along m ; and select the one corresponding to the last crossing c_s . Let $S_2 \subset S_1$ be the set of selected end segments, we have $|S_2| \geq \frac{1}{k}|S_1| \geq \frac{1}{k}|E_1|$.

Drawing red edges. We construct a directed multi-graph $G' = (V, E')$. We call the edges in E' *red*, to distinguish them from the edges of E . For every end segment $s \in S_2$, we construct a red edge $\gamma(s)$, which is a polygonal path with two bends between the vertex u_s incident to s and another vertex in V . The red edge $\gamma(s)$ is constructed as follows. Recall that c_s is the first crossing of s with some middle segment, say m_s . Let d_s be the next crossing along m_s (following the direction of m_s), or if c_s is the last crossing along m_s , then let d_s be the endpoint of m_s . In both cases, d_s lies on a unique end segment, which is incident to a unique vertex $v_s \in V$. Now $\gamma(s)$ is the directed path that follows segment s from u_s to c_s ; it follows the middle segment m_s from c_s to d_s ; and finally follows another end segment from d_s to v_s . See Figure 4(a) for an example. Since we assume that G is drawn with the minimal number of crossings, we have $u_s \neq v_s$.

Indeed, suppose that $u_s = v_s$. If d_s is the endpoint of the middle segment m_s , then we could redraw the edge e_m containing m_s so that edge e_m ends right before reaching point c_s and then continues to u_s closely following along s without crossings. Observe that by this redrawing we reduce the number of crossings without violating conditions (1)–(3). Otherwise d_s is a crossing of m_s with an end segment, say s' , which is incident to $u_s = v_s$. In this case, we could redraw the part of s' between v_s and d_s so that it closely follows s from v_s to c_s and then follows m_s to d_s without crossing any edge. Since $d_s \neq c_{s'}$, the end segment s' crosses some middle segment before d_s , and so this redrawing reduces the number of crossings without violating conditions (1)–(3).

Every red edge $\gamma(s)$ is naturally partitioned into two end segments, (u_s, c_s) and (d_s, v_s) , and one middle segment (c_s, d_s) . The first segment of a red edge $\gamma(s)$ lies along the end segment $s \in S_2$, and it is crossing-free. The middle segment of $\gamma(s)$ lies along a middle segment m_s , follows the direction of m_s , and it is also crossing-free. The last segment of $\gamma(s)$ lies along an end segment, and possibly

Fig. 4. Illustrations for the proof of Theorem 5

have a common tail with other red edges. Note that the last segment of $\gamma(e)$ does not cross any other red edges, since all crossings in G are between end segments and middle segments, and by construction, the middle segments of red edges do not cross any end segments. Note also that two red edges cannot follow the same polygonal path in opposite directions (e.g., (u, v) and (v, u)), since every red edge follows a prescribed direction along its middle segment. If two red edges overlap, then the overlap lies on their end segments, and they have a common tail. Thus, they can be removed using Lemma 3, and we obtain a planar multi-graph.

Recall that a planar multi-graph on n vertices has at most $3n$ edges, if it has no faces of size 2. We show that by removing at most a $\frac{4k+3}{4k+4}$ fraction of the edges of G' , the remaining planar multi-graph will have no 2-faces.

Proposition 1. *Let $\gamma_1, \gamma_2, \gamma_3$ be three parallel red edges from u to v in G' . Let A be the region bounded by γ_1 and γ_3 and containing γ_2 . Let $e_2 \in E$ be the edge that contains the middle segment of γ_2 , and let s_2 be the middle segment of e_2 . If e_2 has an endpoint $w \neq u, v$ (that is, $e_2 \neq (u, v)$), then $w \in A$ or s_2 crosses the last segment of γ_1 or γ_3 . (See Figure 4(b).)*

Proof. By construction, only the last segment of a red edge can cross any edge of the original graph G . Hence if e_2 has an endpoint $w \neq u, v$ and $w \notin A$, then s_2 must cross the last segment of γ_1 or γ_3 .

For every pair of vertices u, v we consider the two sets of red edges (u, v) and (v, u) , and remove the smaller one. Note that at most half of the red edges were deleted. Let $\gamma_1, \gamma_2, \dots, \gamma_{2k+4}$ be parallel red edges from u to v in G' , listed according to their cyclic order around u . Denote by A the region bounded by γ_1 and γ_{2k+4} , and containing the rest of the edges. Let s_i be the middle segment in G that contains the middle segment of γ_i , for $i = 1, 2, \dots, 2k+4$, and let e_i be the edge of G that contains s_i . Since G is a simple graph, one of e_{k+2} and e_{k+3} is

not an edge between u and v . Assume, w.l.o.g., it is e_{k+2} . Applying Proposition 1 on e_{k+2} and every pair of edges e_i, e_{2k+5-i} , $i = 1, 2, \dots, k+1$, we conclude that there must be a vertex $w \neq u, v$ of e_{k+2} in A , or s_{k+2} crosses at least $k+1$ end segments that are adjacent to v . However, the latter cannot happen since every middle segment in G crosses by at most k end segments that share a vertex.

Therefore, by deleting all but one of every $2k+3$ consecutive parallel red edges around a vertex, we obtain a planar graph with no 2-faces. In the worst case, we have $4k+4$ consecutive parallel edges between u and v , and keep only one. By removing at most $(4k+3)/(4k+4)$ fraction of the edges in G' , we obtain a planar multi-graph with no 2-faces, and thus having at most $3n$ edges. It follows that the number of red edges is bounded by

$$\frac{1}{2}|E'| \leq (4k+4)3n = 12(k+1)n.$$

Since $|E'| \geq \frac{1}{k}|E_1|$, it follows that $|E| < |E_1| + 3n \leq k|E'| + 3n \leq 24(k+1)kn + 3n$. $\square$

3.3 Proofs of Theorems 1 and 3

We are now ready to prove that a graph on n vertices that admits an αAC_2^- drawing has $O(n)$ edges.

Proof of Theorem 1: Let $G = (V, E)$ be a graph with n vertices drawn in the plane with an αAC_2^- drawing. It follows from Lemma 4 that the number of edges in G which have end segments crossing some other end segments, is at most $36n$. Let $G_1 = (V, E_1)$ be the subgraph of G that does not contain two mutually crossing end segments.

Let S_1 be the set of *middle* segments of all edges in E_1 . By Lemma 1, there is a subset $S_2 \subset S_1$ of at least $\frac{1}{3}|S_1| = \frac{1}{3}|E_1|$ pairwise noncrossing segments. Let $E_2 \subseteq E_1$ be the set of edges whose middle segments are in S_2 , and let $G_2 = (V, E_2)$ be a subgraph of G with at least $\frac{1}{3}|E_1|$ edges.

Note that G_2 has properties (1)–(3) with $k = 2$ in Theorem 5. All crossings are between an end segment and a middle segment. Moreover, no middle segment crosses more than two end segments that share a vertex, since two such end segments form an isosceles triangle with the middle segment.

Therefore, it follows from Theorem 5 that the number of edges in E_2 is at most $147n$. Therefore, G has at most $36n + 3 \cdot 147n = 477n$ edges. $\square$

With a slightly more effort one can prove Theorem 3.

Proof of Theorem 3: Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. Let G be a graph on n vertices that admits a drawing with at most two bends per edge such that every crossing occurs at some angle from A .

Using an easy generalization of Lemma 1 (c.f., [4]), G has a subgraph with at least $|E|/(2k+1)$ edges that does not contain two crossing middle segments. This graph has at most $3n$ edges that have crossing-free end segments. Note that a segment can be crossed by at most $2k$ end segments that share a vertex. Therefore,

The number of edges whose end segments are only crossed by middle segments is at most $O(k^2n)$ by Theorem 5. It remains to count the number of edges that have an end segment which is crossed by another end segment. By Lemma 5, there are at most $O(kn)$ such edges. Therefore, G has $O(k^3n)$ edges. $\square$

4 Proof of Theorem 4

We prove Theorem 4 using the same ideas as in the proofs of Lemma 4 and Theorem 5.

Proof of Theorem 4: Let G be a topological graph on n vertices, and suppose that every edge of G is partitioned into a red end segment and a blue end segment, such that: (1) no two end segments of the same color cross; (2) every pair of end segments intersects at most once; and (3) no end segment is crossed by more than k end segments that share a vertex. We show that G has $O(kn)$ edges.

By [1] the graph G has at most $6.5n - 20$ edges with a crossing free end segment, since these edges form a simple quasi-planar graph. Denote the rest of the edges by E_1 . For every edge $e = (u, v) \in E_1$ we draw a new edge $\gamma(e)$ as follows. Let (u, x) denote the crossed red end segment of e . Direct (u, x) from u to x , and let y be the first crossing point on it. Let e' be the edge that crosses e at y . Then y must lie on the blue end segment of e' . Denote by w the vertex that is adjacent to the blue end segment of e' , and observe that $w \neq u$, for otherwise e and e' have end segments that intersect twice. Define $\gamma(e) = (u, y) \cup (y, w)$, orient it from u to w , and call it a *red-blue* edge.

For every pair of vertices u, v , we consider the two sets of parallel edges from u to v and parallel edges from v to u . We remove the smaller set. Denote the resulting multi-graph by $G' = (V, E')$, and observe that $|E'| \geq |E_1|/2$. Note also that G' is a plane graph. Indeed, crossings may occur only between a red end segment and a blue end segment, however, the red end segment of every edge in G' is crossing-free. G' might contain edges with a common tail, however, these overlaps may be removed using Lemma 3.

Proposition 2. *Let $\gamma_1, \gamma_2, \gamma_3$, be three parallel red-blue edges from u to w in G' . Let A be the region bounded by γ_1 and γ_3 and containing γ_2 . Let r_2 be the red end segment of G that contains the red end segment of γ_2 , and let e_2 be the edge of G that contains r_2 . Then if $e_2 = (u, v)$ for $v \neq w$ then either v lies inside A or r_2 crosses the blue end segment of γ_1 or γ_3 .*

Proof. Suppose that $e_2 = (u, v)$ such that $v \neq w$. If $v \notin A$ then e_2 must cross γ_1 or γ_3 . Since crossings occur only between red and blue end segments, and the red end segment of a red-blue edge is crossing-free, it follows that r_2 must cross the blue end segment of γ_1 or γ_3 .

Let $\gamma_1, \gamma_2, \dots, \gamma_{2k+4}$ be parallel red-blue edges from u to w in G' , listed according to their cyclic order around u . Denote by A the region bounded by γ_1 and γ_{2k+4} , and containing the rest of the edges. Let r_i be the red end segment of G that

contains the red end segment of γ_i , $i = 1, \dots, 2k + 4$, and let e_i be the edge of G that contains r_i . One of e_{k+2}, e_{k+3} is not an edge between u and w . Assume, w.l.o.g., it is e_{k+2} . Applying Proposition 2 on γ_{k+2} and every pair of edges $\gamma_i, \gamma_{2k+5-i}$, $i = 1, 2, \dots, k + 1$, we conclude that there must be a vertex $v \neq u, w$ of e_{k+2} in A , or r_{k+2} crosses at least $k + 1$ blue end segment that are adjacent to w . However, the latter cannot happen since every end segment in G is crossed by at most k end segments that share a common vertex.

Therefore, by keeping one out of every $2k + 3$ consecutive parallel red edges around a vertex, we obtain a graph with no 2-faces. In the worst case, we keep only one edge out of $4k + 4$ parallel edges. Therefore, by removing at most $(4k + 3)/(4k + 4)$ of the edges in G' we obtain a planar multi-graph with no 2-faces and therefore with at most $3n$ edges. We conclude that $|E| \leq |E_1| + 6.5n \leq 2|E'| + 6.5n \leq 2(4k + 4)3n + 6.5n = (24k + 30.5)n$. $\square$

Remark. It follows from Theorem 4 that a graph on n vertices that admits an αAC_1^- drawing has $O(n)$ edges (however, with a worse constant coefficient than in Theorem 2): Let $\alpha \in (0, \frac{\pi}{2}]$ and let $G = (V, E)$ be a graph on n vertices that admits an αAC_1^- drawing. We first partition arbitrarily every straight-line edge of G into two end segments. Next, for every edge of G we assign the color red to one of its end segments, and the color blue to its other end segment. Applying Lemma 1 twice we conclude that there is a set of edges $E' \subseteq E$, such that no two end segments of the same color of two edges in E' cross, and $|E'| \geq |E|/9$. Recall that an end segment in G cannot be crossed by more than two end segments that share a common vertex. Therefore, by Theorem 4 we have that $|E'| \leq 78.5n$ and thus $|E| \leq 706.5n$.

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