A Type of Arithmetic Labels about Circulating Ring

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Abstract. The graph composed with several rings is a kind of important and interesting graph, many scholars studied on the gracefulness of this kind of the graph, The reference [1] is given the gracefulness of m kinds C_4 with one common point. In this paper, we researched the arithmetic labels of four kinds graph: $C_{8,1,n}$, $C_{8,2,n}$, $C_{8,3,n}$, $C_{8,4,n}$, and we proved they are all $(d, 2d)$ --arithmetic graph.

Keywords: Arithmetic graph, labeling of graph, $C_{8,i,n}$.

1 Introduction

The graph in this paper discussed are undirected, no multiple edges and simple graph, the unorganized state of definitions and terminology and the symbols in this graph referred to reference[2][3].

There are two kinds of labels of the graph: one is the reduced label, is to say that in order to get the label of one edge you should reduce the endpoints of the edge; the other is additive label, for the same you should active the endpoints of one edge to get the label of the edge. For example, the well-known of “Gracefulness” is reduced. the “Compatible labels” is additive. In 1990, B.D.Achaya and S.M.Hegde import the concept of “Arithmetic lables”(referred to reference[2]), which is a more extensive additive label, it have applied value on solution to question of the joint ventures in rights and obligations.

Definition 1.1. For $G = (V, E)$, if there is a mapping f (called the vertices v of the label) from $V(G)$ to the set of nonnegative integer N_0 , meet:

- (1) $f(u) \neq f(v)$, which $u \neq v$, and $u, v \in V(G)$;
- (2) $\{f(u) + f(v) | uv \in E(G)\} = \{k, k + d, \dots, k + (q - 1)d\}$.

Then we call graph G is a (k, d) --arithmetic graph.

2 Main Results and Certification

Theorem 2.1. $C_{8,1,n}$ is a $(d, 2d)$ -arithmetic graph.

Proof. As the graph shown on fig.1.

Fig. 1. The Graf $C_{8,1,n}$

Label all vertices as follows:

$$\begin{aligned} f(v_1^{(i)}) &= 8id \ (i = 0, 1, 2, \dots, n), & f(v_2^{(i)}) &= 8id + d \ (i = 0, 1, 2, \dots, n-1), \\ f(v_3^{(i)}) &= 8id + 3d \ (i = 0, 1, 2, \dots, n-1), & f(v_4^{(i)}) &= 8id + 6d \ (i = 0, 1, 2, \dots, n-1), \\ f(v_5^{(i)}) &= 8id + 2d \ (i = 0, 1, 2, \dots, n-1), & f(v_6^{(i)}) &= 8id + 5d \ (i = 0, 1, 2, \dots, n-1), \\ f(v_7^{(i)}) &= 8id + 7d \ (i = 0, 1, 2, \dots, n-1). \end{aligned}$$

Now we proof that the mapping f is arithmetic labeling of $C_{8,1,n}$.

We can see the mapping f meet $f(u) \neq f(v)$ which $u \neq v$ and $u, v \in V(C_{8,1,n})$.

Next we prove $\{f(u) + f(v) \mid uv \in E(C_{8,1,n})\}$ is an arithmetic progression in the way of mathematical induction.

When $n = 1$

Then $f(v_1^{(0)}) = 0$, $f(v_2^{(0)}) = d$, $f(v_3^{(0)}) = 3d$, $f(v_4^{(0)}) = 6d$, $f(v_5^{(0)}) = 2d$, $f(v_6^{(0)}) = 5d$, $f(v_7^{(0)}) = 7d$, $f(v_1^{(1)}) = 8d$.

Therefore $\{f(u) + f(v) \mid uv \in E(C_{8,1,1})\}$

$$\begin{aligned} &= \{f(v_1^{(0)}) + f(v_2^{(0)}), f(v_1^{(0)}) + f(v_3^{(0)}), f(v_3^{(0)}) + f(v_5^{(0)}), \\ &f(v_2^{(0)}) + f(v_4^{(0)}), f(v_5^{(0)}) + f(v_7^{(0)}), f(v_4^{(0)}) + f(v_6^{(0)}), f(v_6^{(0)}) + f(v_1^{(1)}), \\ &f(v_7^{(0)}) + f(v_1^{(1)})\} \\ &= \{d, 3d, 5d, 7d, 9d, 11d, 13d, 15d\} \end{aligned}$$

is an arithmetic progression, and the common difference is $2d$.

Suppose when $n = k - 1$, we know

$$\{f(u) + f(v) | uv \in E(C_{8,1,k-1})\} = \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (8k - 9) \times 2d\}$$

is an arithmetic progression, and the common difference is $2d$.

Then when $n = k$

$$\begin{aligned} & \{f(u) + f(v) | uv \in E(C_{8,1,k})\} \\ &= \{f(u) + f(v) | uv \in E(C_{8,1,k-1})\} \cup \{f(v_1^{(k-1)}) + f(v_2^{(k-1)}), f(v_1^{(k-1)}) + f(v_3^{(k-1)}), \\ & f(v_3^{(k-1)}) + f(v_5^{(k-1)}), f(v_2^{(k-1)}) + f(v_4^{(k-1)}), f(v_5^{(k-1)}) + f(v_7^{(k-1)}), \\ & f(v_4^{(k-1)}) + f(v_6^{(k-1)}), f(v_6^{(k-1)}) + f(v_1^{(k)}), f(v_7^{(k-1)}) + f(v_1^{(k)})\} \\ &= \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (8k - 9) \times 2d\} \cup \{d + (8k - 8) \times 2d, \\ & d + (8k - 7) \times 2d, d + (8k - 6) \times 2d, d + (8k - 5) \times 2d, d + (8k - 4) \times 2d, \\ & d + (8k - 3) \times 2d, d + (8k - 2) \times 2d, d + (8k - 1) \times 2d\} \\ &= \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (8k - 1) \times 2d\} \end{aligned}$$

is an arithmetic progression, and the common difference is $2d$.

In sum for the arbitrary $n \in N_0$, the mapping $f : V(C_{8,1,n}) \rightarrow N_0$ meet:

- (1) $f(u) \neq f(v)$ when $u \neq v$ and $u, v \in V(C_{8,1,n})$;
- (2) $\{f(u) + f(v) | uv \in E(C_{8,1,n})\} = \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (8n - 1) \times 2d\}$.

Therefore $C_{8,1,n}$ is a $(d, 2d)$ --arithmetic graph.

Theorem 2.2. $C_{8,2,n}$ is a $(d, 2d)$ --arithmetic graph.

Proof. As the graph shown on fig.2.

Fig. 2. The Graf $C_{8,2,n}$

Label all vertices as follows:

$$\begin{aligned} f(v_1^{(i)}) &= 7id \ (i=0,1,2,\dots,n), & f(v_4^{(i)}) &= 7id + 6d \ (i=0,1,2,\dots,n-1), \\ f(v_2^{(i)}) &= 7id + d \ (i=0,1,2,\dots,n), & f(v_5^{(i)}) &= 7id + 2d \ (i=0,1,2,\dots,n-1), \\ f(v_3^{(i)}) &= 7id + 3d \ (i=0,1,2,\dots,n-1), & f(v_6^{(i)}) &= 7id + 5d \ (i=0,1,2,\dots,n-1). \end{aligned}$$

Now we proof that the mapping f is arithmetic labeling of $C_{8,2,n}$

We can see the mapping f meet $f(u) \neq f(v)$ which $u \neq v$ and $u, v \in V(C_{8,2,n})$.

Next we prove $\{f(u) + f(v) | uv \in E(C_{8,2,n})\}$ is an arithmetic progression in the way of mathematical induction.

When $n = 1$

Then $f(v_1^{(0)}) = 0$, $f(v_2^{(0)}) = d$, $f(v_3^{(0)}) = 3d$, $f(v_4^{(0)}) = 6d$, $f(v_5^{(0)}) = 2d$, $f(v_6^{(0)}) = 5d$, $f(v_1^{(1)}) = 7d$, $f(v_2^{(1)}) = 8d$.

$$\begin{aligned} \text{Therefore } \{f(u) + f(v) | uv \in E(C_{8,2,1})\} \\ &= \{f(v_1^{(0)}) + f(v_2^{(0)}), f(v_1^{(0)}) + f(v_3^{(0)}), f(v_3^{(0)}) + f(v_5^{(0)}), \\ &f(v_2^{(0)}) + f(v_4^{(0)}), f(v_5^{(0)}) + f(v_1^{(1)}), f(v_4^{(0)}) + f(v_6^{(0)}), f(v_6^{(0)}) + f(v_2^{(1)}), \\ &f(v_1^{(1)}) + f(v_2^{(1)})\} \\ &= \{d, 3d, 5d, 7d, 9d, 11d, 13d, 15d\} \end{aligned}$$

is an arithmetic progression, and the common difference is $2d$.

Suppose when $n = k - 1$, we know

$$\{f(u) + f(v) | uv \in E(C_{8,2,k-1})\} = \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (7k - 7) \times 2d\}$$

is an arithmetic progression, and the common difference is $2d$.

Then when $n = k$

$$\begin{aligned} \{f(u) + f(v) | uv \in E(C_{8,2,k})\} \\ &= \{f(u) + f(v) | uv \in E(C_{8,2,k-1})\} \cup \{f(v_1^{(k-1)}) + f(v_3^{(k-1)}), \\ &f(v_3^{(k-1)}) + f(v_5^{(k-1)}), f(v_2^{(k-1)}) + f(v_4^{(k-1)}), f(v_5^{(k-1)}) + f(v_1^{(k)}), \\ &f(v_4^{(k-1)}) + f(v_6^{(k-1)}), f(v_6^{(k-1)}) + f(v_2^{(k)}), f(v_1^{(k)}) + f(v_2^{(k)})\} \end{aligned}$$

$$\begin{aligned}
 &= \{d, d+1 \times 2d, d+2 \times 2d, \dots, d+(7k-7) \times 2d\} \cup \{d+(7k-6) \times 2d, \\
 &d+(7k-5) \times 2d, d+(7k-4) \times 2d, d+(7k-3) \times 2d, d+(7k-2) \times 2d, \\
 &d+(7k-1) \times 2d, d+7k \times 2d\} \\
 &= \{d, d+1 \times 2d, d+2 \times 2d, \dots, d+7k \times 2d\}
 \end{aligned}$$

is an arithmetic progression, and the common difference is $2d$.

In sum for the arbitrary $n \in N_0$, the mapping $f : V(C_{8,2,n}) \rightarrow N_0$ meet:

- (1) $f(u) \neq f(v)$ when $u \neq v$ and $u, v \in V(C_{8,2,n})$;
- (2) $\{f(u) + f(v) \mid uv \in E(C_{8,2,n})\} = \{d, d+1 \times 2d, d+2 \times 2d, \dots, d+7n \times 2d\}$.

Therefore $C_{8,2,n}$ is a $(d, 2d)$ --arithmetic graph.

Theorem 2.3. $C_{8,3,n}$ is a $(d, 2d)$ --arithmetic graph.

Proof. As the graph shown on fig.3.

Fig. 3. The Graf $C_{8,3,n}$

Label all vertices as follows:

$$\begin{aligned}
 f(v_0^{(i)}) &= 8id \ (i = 0, 1, 2, \dots, n), & f(v_1^{(i)}) &= 4id + d \ (i = 0, 1, 2, \dots, n), \\
 f(v_3^{(i)}) &= 4id + 3d \ (i = 0, 1, 2, \dots, n), & f(v_2^{(i)}) &= 8id + 6d \ (i = 0, 1, 2, \dots, n-1), \\
 f(v_4^{(i)}) &= 8id + 2d \ (i = 0, 1, 2, \dots, n-1).
 \end{aligned}$$

Proof in imitation of Theorem 2.1.

For the arbitrary $n \in N_0$, the mapping $f : V(C_{8,3,n}) \rightarrow N_0$ meet:

- (1) $f(u) \neq f(v)$ when $u \neq v$ and $u, v \in V(C_{8,3,n})$;
- (2) $\{f(u) + f(v) \mid uv \in E(C_{8,3,n})\} = \{d, d+1 \times 2d, d+2 \times 2d, \dots, d+(6n+1) \times 2d\}$.

By (1), (2) we know the mapping f is arithmetic labeling of $C_{8,3,n}$. Therefore $C_{8,3,n}$ is a $(d, 2d)$ --arithmetic graph.

Theorem 2.4. $C_{8,4,n}$ is a $(d, 2d)$ --arithmetic graph.

Proof. As the graph shown on fig.4.

Fig. 4. The Graf $C_{8,4,n}$

Label all vertices as follows:

$$f(v_1^{(i)}) = 5id \ (i = 0, 1, 2, \dots, n), \quad f(v_2^{(i)}) = 5id + d \ (i = 0, 1, 2, \dots, n),$$

$$f(v_3^{(i)}) = 5id + 3d \ (i = 0, 1, 2, \dots, n), \quad f(v_4^{(i)}) = 5id + 2d \ (i = 0, 1, 2, \dots, n).$$

Proof in imitation of Theorem 2.2.

For the arbitrary $n \in N_0$, the mapping $f : V(C_{8,4,n}) \rightarrow N_0$ meet:

- (1) $f(u) \neq f(v)$ when $u \neq v$ and $u, v \in V(C_{8,4,n})$;
- (2) $\{f(u) + f(v) \mid uv \in E(C_{8,4,n})\} = \{d, d + 1 \times 2d, d + 2 \times 2d, \dots, d + (5n + 2) \times 2d\}$.

By (1), (2) we know the mapping f is arithmetic labeling of $C_{8,4,n}$. Therefore $C_{8,4,n}$ is a $(d, 2d)$ --arithmetic graph.

3 Labels of Some Special Graph

In order to explain the correctness of the aforementioned labels, we give the arithmetic labeling of $C_{8,1,3}$, $C_{8,2,3}$, $C_{8,3,3}$ and $C_{8,4,3}$.

(1) The arithmetic labeling of $C_{8,1,3}$ on fig.5.

Fig. 5. The Graf $C_{8,1,3}$

(2) The arithmetic labeling of $C_{8,2,3}$ on fig.6.

Fig. 6. The Graf $C_{8,2,3}$

(3) The arithmetic labeling of $C_{8,3,3}$ on fig.7.

Fig. 7. The Graf $C_{8,3,3}$

(4) The arithmetic labeling of $C_{8,4,3}$ on fig.8.

Fig. 8. The Graf $C_{8,4,3}$

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