Efficient Time and Frequency Methods for Sampling Filter Functions

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Abstract. In this paper, we seek to determine the adequate number of samples for an analog filter function $f(t)$. The proposed approaches provide discrete filters that can be used for multiresolution analysis. We introduce two methods that provide sampling results for localization: one of them defines an approximate Nyquist rate, and the other samples in a manner that ensures time-frequency consistency between the generated samples and the analog filter function. The key contribution of the paper is that it establishes robust mathematical and programmable foundations for a previously established empirical method. Analytically, we show that the time-frequency method is based on minimizing aliasing while maximizing decimation. The method can be programmed by introducing a mean square error (MSE) threshold across scales. Afterwards, we provide the outcomes of experiments that demonstrate success of localization with the proposed time-frequency method.

Keywords: Multiresolution analysis; multiscale analysis; sampling filters; time-frequency.

1 Introduction

Sampling is a key step in digital signal processing for transforming a mathematical continuous function into a discrete function for digital processing. In this paper, we examine the problem of sampling a continuous filter function to derive samples for multi-resolution analysis. Several researchers [1-9] have faced this step in their multiscale applications. Some of the work does not clearly describe how the samples were derived, while others included restriction conditions to ensure perfect function reconstruction for the samples.

There has been extensive research and publications in sampling methods for multiresolution analysis. The development of wavelets by Mallat [10,11] as filters for multiresolution, and the establishment of orthonormal basis with compact support by Daubechies [12, 13] are considered key foundation references for multiresolution sampling theory and analysis. Applications in multiresolution analysis have at times relied on analog filters as they provide an elegant mathematical derivation, and avoided the sampling problem [9].

Level-crossing has contributed to the literature of sampling signals on one hand and of feature detection on the other. Guan and Singer [14] proposed sampling by

level-crossings based on the Lebesgue integral that approximates frequency non-bandlimited functions with a finite set of amplitude. Despite its advantages, this divides the amplitude of the signal into equally spaced intervals in order to set the levels of the crossings, whereas it does not specify the number of levels that should be taken nor does it indicate how to obtain this number.

Much research has been done on analog-to-digital transforms, or s-to-z, transforms. The approaches for these transforms, according to [15], can be divided into three categories: time-invariant response methods, numerical approximation methods, and heuristic methods. These transforms form a basis for mapping an analog filter to a digital one. However, in this mapping, these methods still require a specification of how to select the sampling period T in order to finalize all aspects of this transformation.

Shannon sampling provides a robust way for sampling bandlimited signals; however, most filter functions we deal with are time bandlimited and thus non-bandlimited in frequency, and an alternative approach is needed in these areas. Several approaches have been proposed, each however posing its own limitations. Liu et al. [16] establish a generalized Shannon-type sampling theorem for a class of non-bandlimited signals with special spectrum properties regarding a ladder shaped filter of two parameters. Moreover, Chen et al [17] successfully propose a theorem to reconstruct certain non-bandlimited signals; they also provide several sampling formulae that remain challenged by the question of identifying six relevant parameters required to sample. In short, the question of sampling different classes of non-bandlimited signals remains open in many respects.

Accordingly, the remaining of the paper is organized as follows. Section 2 proposes a method to approximate Nyquist rate for filter signals which are time-bandlimited, and thus frequency non-bandlimited. Section 3 introduces the time-frequency method which ensures consistency in both time and frequency domains by constructing a discrete time-consistent model and provides the mathematical foundation and evaluation for this method followed by evaluation and comparison in section 4. Section 5, then, summarizes the results and section 6 concludes the paper.

2 Frequency-Based Method

This section of the paper presents a method to sample non-frequency-bandlimited signals by employing continuous and discrete time signal analysis. This is analogous to finding the bounds of an anti-aliasing filter.

Shannon's sampling theorem requires that sampling occurs at a rate at least double the highest frequency of the input signal; thus, it does not address how to sample input functions that having no upper bound on frequency. To sample frequency non-bandlimited functions, we draw from Parseval's theorem equivalence of energy between time and frequency. Unlike bandlimited functions whose integral in frequency domain is bounded by the highest frequency, we must approximate the boundary frequency of non-bandlimited functions. Our approximation employs a threshold ratio of the total energy to derive the sampling rate needed to produce an output signal that contains a percentage of the energy content of the total input function, where 1 i.e. 100% represents the whole energy.

2.1 Procedure

Given an approximation ratio, e.g. 0.9 to represent 90% of energy approximation, the procedure for determining the upper frequency bound is described below. First, consider the filter continuous function $f(t)$, and derive its Fourier transform $F(w)$. Then, we calculate its total energy content. Finally, we solve the equation

$$\frac{1}{2\pi} \int_{-W_h}^{W_h} |F(w)|^2 dw = \tau * E_{total} = \tau * \int_{-\infty}^{\infty} |f(t)|^2 dt . \tag{1}$$

Solving the above equation means finding the frequency w_{high} (W_h), at which the energy content of the filter signal will have reached $\tau*100\%$ of its total value. Now, the approximation of Nyquist theorem implies that the sampling should take place at a frequency at least equal to twice W_h . Therefore, the frequency required is

$$w_{sampling} = \lceil 2 * W_h \rceil . \tag{2}$$

2.2 Examples

To illustrate this procedure, we consider some filter functions namely Haar, Gaussian, First Derivative of Gaussian (FOG), Laplacian of Gaussian (LOG), and Pulse, and apply the method with $\tau = 0.9$ to derive the required sampling rates. The results are summarized in Table 1. For the Haar and the Pulse filter functions, the time frame considered was 2 seconds by design. The Gaussian, on the other hand, has a standard deviation $\sigma=1$ sec; therefore, it is safe to consider $6\sigma = 6$ sec as a complete time support for the function. This time support was also adopted for the FOG and LOG.

Table 1. Required Number of Samples Using the Frequency-based method

<i>Filter Function</i>	<i>Number of samples needed</i>
Haar	40
Gaussian	24
First Derivative of Gaussian (FOG)	92
Laplacian of Gaussian (LOG)	30
Pulse	32

3 Time Frequency Consistency

Our approach of sampling filter functions at lower rates than Nyquist aims at modeling these filters using smaller supports without losing their practical representations. The key objective is that the filter samples would provide us best localization when applied to signals. This section describes a method that builds on the time and frequency consistencies (TFC) across scales. In [18], Hajj relied on visual inspector of the Z-transform. In this paper, we develop an analytical and programmable approach to eliminate the required visual spectrum. The essence of the method is to maintain consistency in both time and frequency domains, which would thus yield a representation of filter functions at lowest sampling rate, yet which are consistent with their representations in both time and frequency analysis, and thus

may be used to provide viable models for the filter samples. On one hand, maintaining consistency in time is achieved by construction of the samples which are chosen in discrete time in a manner that roughly represents the continuous time model. Then, we proceed by finding the Z-transform of a filter signal at lowest scale (i.e. at 2 or 3 samples), whose magnitude is to be plotted over $[0, \pi]$. The number of samples taken afterwards is then increased dyadically until the Z-transform shows consistency in the plot. Starting to have consistency at a certain level implies that the corresponding number of samples is the sufficient representation of the function in the frequency domain as will be demonstrated in the following subsection. Determining a sufficient representation of the samples may be assumed when the following two criteria are satisfied:

1. The Z-transform is evident within $[0, \pi]$, i.e. the whole shape of the transform lies inside $[0, \pi]$ not outside it.
2. The transform plot has sufficient decay to 0, i.e. according to a threshold pre-specified (90% for example).

3.1 Programmable Approach

This section illustrates the results of TFC thus verifying its applicability from a numerical perspective before resolving to its proof from a mathematical viewpoint.

To automate the sampling procedure and satisfy the above two conditions, the following procedure is implemented. First, starting at 2 or 3 samples, we construct the magnitude of the Z-transform evenly spaced between 0 & π . Then, at any scale 'a' we consider the following normalized sum:

$$e_a = \sqrt{\frac{2}{N} \sum_{i=1}^{N/2} (|Z_a[i]| - |Z_{a-1}[2i]|)^2}. \quad (3)$$

where N is the number of points in computing the Z-transform and $|Z_a[i]|$ is the magnitude of the Z-transform at a scale a and point i. This sum is the normalized error expressed as the root-mean-square (e_a) of the errors between two successive scales.

In our experiments, we found that a threshold of 16% was sufficient for achieving TFC. However, we must also maintain that this error should apply across all scales, which can be detected by having the threshold satisfied at a pair of scales then having decreasing errors from one scale to the next.

Programmatically, thus, we may now formulate the conditions, in order to consider scale 'a' as sufficient, we must have:

1. MSE between scales: $e_a^2 < 0.16$
2. If, at one level, the error meets this threshold requirement, we must also check whether the following error (i.e. that of the next transition) is less than the former.

If condition 2 is not satisfied, we would not be able to ensure consistency at higher scales.

In order to verify this method, it is tested on several filter signals, namely: Haar, Gaussian and its first two derivatives, roof, triangle, and pulse, and the results are shown in table 2.

Table 2. Required Number of Samples Using TFC

<i>Filter Function</i>	<i>Number of Scales Examined</i>	<i>Consecutive e_a^2's</i>	<i>Resulting number of samples needed</i>
Haar	4	0.4941, 0.2479, 0.1229, 0.0604	8 (level 3)
Gaussian	4	0.4202, 0.2153, 0.0972, 0.0476	9 (level 3)
First Derivative of Gaussian (FOG)	5	3.4606e-005, 0.1924, 0.2575, 0.1100	17 (level 4)
Laplacian of Gaussian (LOG)	5	0.5808, 0.3672, 0.4517, 0.1654, 0.0764	17 (level 4)
Roof	3	0.1061, 0.0813	5 (level 2)
Triangle	5	2.9052e-005, 0.5208, 0.1826, 0.0802, 0.0375	17 (level 4)
Pulse	4	0.1229, 0.4176, 0.1365, 0.0970	9 (level 3)

3.2 Analytic Approach

Having illustrated an automated programmable method of TFC in sampling filter functions, we provide the mathematical foundation of the method along with an interpretation of the results obtained. Similar to the analysis of the thresholding operator in [18], we can show that our TFC sampling operator is non-linear and discontinuous, which implies that it is difficult to analyze. Nevertheless, we demonstrate that the foundation of the method is an optimization problem based on minimizing aliasing and maximizing decimation.

3.2.1 Sampling Consistency in Time Domain

Often, when we need a larger support for a filter function, the approach is to upsample this signal (function) by N , then filter the output signal to smooth out the filter. The derivation of the upsampling of a signal $x[n]$:

$$y[n] = x^{(N)}[n] = \begin{cases} x\left[\frac{n}{N}\right] & \text{if } n \text{ is a multiple of } N \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

results in the following corresponding frequency relation [19]

$$Y(z) = X^{(N)}(z) = X(z^N). \quad (5)$$

However, this refined filter introduces aliasing, or “inaccurate” values at the skipped sampling points. On the other hand, TFC provides us with the ability to widen the support of our filter signal by adding exact samples at the skipped sampling points rather than introducing these “inaccurate” samples

Many systems are aware of the filter functions in the time domain. In such cases, TFC supplies these systems with a much better approximation for the discrete filter functions, yielding exact samples at the sampling instants, as opposed to approximated samples at half the sampling intervals in the conventional method even at the first step of upsampling by 2. Note that all we need to do in TFC is to generate these new samples from the provided filter function while keeping the old ones.

3.2.2 Sampling Consistency in Frequency Domain

The literature [20] explains how downsampling a signal $x[n]$ by M :

$$y[n] = x[nM]$$

yields the following relation in the frequency domain:

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi k}{M}}\right). \quad (6)$$

This perspective allows us to explain two observed phenomena in TFC:

- The expansion of the Z-transform induced by the downsampling derivation explains the shrinking effect of the TFC frequency plots when comparing to higher scales.
- In downsampling, if $x[m]$ is not bandlimited to n/M , aliasing may result from spectral overlap. Thus, as we downsample, if a signal is bandlimited, increasing M to a certain threshold (defined by the signal's frequency spectrum) starts inducing this aliasing effect. On the other hand, if we consider starting with lowest rates (i.e. number of samples per signal), this aliasing effect is present initially at low rates, and as the rate increases, it decreases until it is completely removed if and only if there exists M such that the corresponding $x[m]$ is bandlimited to n/M in the downsampling perspective. In other words, aliasing can be completely eliminated only if the signal itself is frequency-bandlimited. As a result, to achieve a balance for choosing the smallest set of samples, we choose the samples that give the least aliasing while maximizing decimation. TFC provides the procedure to determine this optimal set of samples and derive larger sets of samples for multiresolution analysis.

4 Advantages of TFC

Having presented the mathematical foundation for TFC, we now present some of the advantages of the proposed method. First of all, like level-crossings [14], in TFC, the signal dictates the frequency of sampling of the filter signal, in our case, even in cases where the signal is frequency non-bandlimited. This feature makes sampling by TFC much simpler from the processing perspective on one hand, while generating minimal number of samples. Moreover, TFC is a parametric sampling scheme with two parameters which can be determined by the sampling system, (1) the MSE (e_a^2) which may be set high for rough edge-detection and low for finer detail-detection, and (2) the bit rate of the sampling process adjustable to a system's limitations and design.

The two advantages discussed so far render TFC a highly scalable sampling approach capable of adapting to multiple purposes and various systems. Furthermore,

to illustrate the advantage of having optimal small sets of samples for multiresolution analysis, consider the following example. If we try to convolve a multi-step signal with a filter, results vary depending on the relative number of samples taken between the filter and the signal. We focus on the Haar filter and the multistep function where step edges can be detected at peaks of the Haar response. From Fig. 1, we can see that TFC responses provide excellent localization while the responses of the filter samples derived from Nyquist approximation have smeared peaks with little localization.

Fig. 1. Signal Convolved with Haar at TFC (left) and Approximated Nyquist (right)

5 Summary of Results

The table below summarizes the sampling rates derived for each of the frequency-based method and TFC for selected filter functions.

Table 3. Number of Samples According to the Three Discussed Methods

Filter function \ Method used	Frequency-based Method (90% threshold)	TFC Method (# of samples needed)
Haar	40	8
LOG	30	17
FOG	92	17
Gaussian	24	9
Pulse	32	9

The table clearly shows the benefit of using the two proposed methods in reducing the number of samples needed from infinite to finite amounts for filter functions that decay to zero in the frequency domain. The relevant restrictions on the properties of the transfer function of such filters' poles and zeroes, regions of convergence, etc. require further analysis and may be within the scope of another paper.

We illustrated how TFC gradually removes aliasing, and how the iterative approach of taking more samples allows us to reach the desired error without overhead. In addition, this approach proves time and resource efficient, allows the filter function to dictate its number of samples, and the application to set the maximum tolerated MSE. Finally, the method has proved superiority in localization. In short, the key contribution of the paper is that it establishes robust mathematical and programmable foundations for a previously established empirical method.

6 Conclusion

For frequency bandlimited signals, Shannon's theorem has provided since long a sufficient condition for reproduction. However, many a time, we encounter filter functions that have a finite time support and thus an infinite frequency representation. Applying the Nyquist criterion for these functions renders us unable to reproduce them given our finite resources of space and time. To overcome this limitation, we provided, in this paper, two methods for filter sampling: the Frequency-based Method and the Time-Frequency Consistency Method. These methods can be applied when our goal isn't the complete reproduction of a filter function, as they use specific forms for signal processing. They also prove beneficial for applications that require few filter samples such as edge localization and multiresolution analysis. We also showed robust programmable and analytical approaches for the TFC method. The smallest set of samples can be derived based on the optimal condition of minimum aliasing and maximum decimation. As these methods are novel, further research could delve more into classifying the types of filters for which our method works best; this will allow us to compare further our methods against recently proposed approaches in compressed sensing, which in turn requires certain properties of the sampling functions.

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