Grey Level Estimation for Discrete Tomography

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Abstract. Discrete tomography is a powerful approach for reconstructing images that contain only a few grey levels from their projections. Most theory and reconstruction algorithms for discrete tomography assume that the values of these grey levels are known in advance. In many practical applications, however, the grey levels are unknown and difficult to estimate. In this paper, we propose a semi-automatic approach for grey level estimation that can be used as a preprocessing step before applying discrete tomography algorithms. We present experimental results on its accuracy in simulation experiments.

1 Introduction

The field of *tomography* deals with the reconstruction of images (known as *tomograms*) from their projections, taken along a range of angles. Tomography has a wide range of applications in medicine, industry, and science. According to [1,2], the field of *discrete tomography* is concerned with the reconstruction of images from a small number of projections, where the set of pixel values is known to have only a few discrete values. It must be noted that the term “*discrete*” is often used to indicate the *discrete domain* of the image, i.e., when reconstructing lattice sets. In this paper, we adhere to the former definition, and focus on the reconstruction of images that consist of a small number of grey levels from their (non-lattice) X-ray projections.

A variety of reconstruction algorithms have been proposed for discrete tomography problems. In [3], an algorithm is presented for reconstructing binary images from a small number of projections. This primal-dual subgradient algorithm is applied to a suitable decomposition of the objective functional, yielding provable convergence to a binary solution. In [4], a similar reconstruction problem is modeled as a series of network flow problems in graphs, that are solved iteratively. Both [5] and [6] consider reconstruction problems that may involve more than two grey levels, employing statistical models based on Gibbs priors for their solution.

Besides the assumption on the grey levels, many of these algorithms incorporate additional prior knowledge of the original object that needs to be reconstructed. A preference for locally smooth regions is typically incorporated, as it

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corresponds to a range of practical images. By combining such prior knowledge with knowledge of the grey levels, it is often possible to compute accurate reconstructions from just a small number of projections. In some cases, as few as three projections is already sufficient to compute a high-quality reconstruction.

A common assumption in all these reconstruction algorithms, is that the set of admissible grey levels is known *a priori*. Although this assumption is easy to satisfy in simulation experiments, obtaining prior knowledge of the set of possible grey levels is often not straightforward in practical applications, for several reasons:

- Even when the number of materials that constitute the scanned object is known, the densities of these materials may be unknown.
- When the materials and their densities *are* known in advance, calibration is required to translate the material properties into a reconstructed grey level. The calibration parameters depend not only on the scanned materials, but also on various properties of the scanner system. In many experimental settings, such information is not available.
- Calibration parameters of the scanning system may change over time. For example, the X-ray source of a micro-CT scanner heats up while scanning, changing the spectrum of emitted X-rays. While this may have a negligible effect on the calibration parameters during a single scan, batch processing of scans may change the parameters, and consequently change the grey levels in the reconstruction.

When several similar objects are scanned as a single batch, it may be possible to obtain a high quality reconstruction of one of those objects, based on a large number of projections. This reconstruction can then be used to estimate the admissible grey levels for the remaining objects, which can then be reconstructed from few projections. However, many important applications of discrete tomography deal with acquisition systems where it is not possible at all to obtain an accurate reconstruction without resorting to discrete tomography. In Materials Science, for example, discrete tomography is used to reconstruct 3D images of nanomaterials from a series of projection images acquired by an electron microscope [7]. As the sample is flat and its thickness increases with the incidence angle of the electron beam, projections can only be obtained for a limited range of angles, resulting in severe reconstruction artifacts for conventional algorithms.

Even in such cases, it may still be possible for an expert user to delineate certain areas in the reconstruction that are likely to have a constant grey level in the reconstruction. In this paper, we therefore consider a simpler version of the grey level estimation problem, where the user first selects an image region that can be expected to contain a constant grey level, based on an initial reconstruction. This reconstruction can be obtained by classical, non-discrete reconstruction algorithms. In certain cases, knowledge of such a constant region allows for reliable estimation of the grey level corresponding to the selected region.

The outline of this paper is as follows. In Section 2, the problem of grey level estimation is introduced, along with formal notation. In Section 3, we present a semi-automatic approach for estimating the grey levels when for each grey

level a subset of corresponding pixels is given. Experimental results for a range of simulation experiments are described in Section 4. Section 5 discusses the possibilities for obtaining more accurate estimates, compared to the approach of this paper. Section 6 concludes the paper.

2 Problem and Model

The unknown physical object from which projection data has been acquired is represented by a grey value image $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We denote the set of all such functions f by $\mathcal{F}$, also called the set of *images*.

Projections are measured along lines $l_{\theta,t} = \{(x,y) \in \mathbb{R}^2 : x \cos \theta + y \sin \theta = t\}$, where θ represents the angle between the line and the y -axis and t represents the coordinate along the projection axis; see Fig. 1.

Denote the set of all functions $s : \mathbb{R} \times [0, 2\pi) \rightarrow \mathbb{R}$ by $\mathcal{S}$. The *Radon transform* $\mathcal{F} \rightarrow \mathcal{S}$ is defined by

$$R(f)(t, \theta) = \iint_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - t) \, dx \, dy.$$

with $\delta(\cdot)$ denoting the Dirac delta function. The function $R(f) \in \mathcal{S}$ is also called the *sinogram* of f .

Fig. 1. Basic setting of transmission tomography

2.1 Grey Level Estimation Problem

The *reconstruction problem* in tomography concerns the recovery of f from its Radon transform $R(f)$. Here, we focus on images f for which the set of grey levels is a small, discrete set $G = \{g_1, \dots, g_k\}$. We denote the set of all such images $f : \mathbb{R}^2 \rightarrow G$ by $\mathcal{G}$. The main problem of this paper consists of estimating the grey levels G from given projection data along a finite set of angles:

Problem 1. Let $D = \{\theta_1, \dots, \theta_d\}$ be a given set of projection angles. Let $G = \{g_1, \dots, g_k\}$ be an unknown set of *grey levels* and let $f \in \mathcal{G}$ be an unknown image. Suppose that k and $R(f)(t, \theta_i)$ are given for $i = 1, \dots, d$, $t \in \mathbb{R}$. Reconstruct G .

It is clear that Problem 1 does not have a unique solution in general. A first requirement for a grey level to be recoverable is that it occurs in f as a measurable subset of $\mathbb{R}^2$. Also, when a small number of projections is available and k is large, it is straightforward to find examples where the grey values cannot be determined uniquely. In this paper, we therefore assume that additional prior knowledge is available, prescribing a set A on which the image is known to have a constant grey level. This leads to the following estimation problem:

Problem 2. Let $D = \{\theta_1, \dots, \theta_d\}$ be a given set of projection angles. Let $G = \{g_1, \dots, g_k\}$ be an unknown set of *grey levels* and let $f \in \mathcal{G}$ be an unknown image. Suppose that k and $R(f)(t, \theta_i)$ are given for $i = 1, \dots, d$, $t \in \mathbb{R}$, and that for $i = 1, \dots, k$, a set $A_i \subset \mathbb{R}^2$ is given such that $f(x, y) = g_i$ for all $(x, y) \in A_i$. Reconstruct G .

However, even specifying an entire region where the image is known to be constant is not sufficient to guarantee uniqueness of the solution of Problem 2. In particular, this problem occurs when the number of projections is small. Fig. 2 shows a well-known procedure to generate a so-called *switching component*: a non-zero image that has a zero projection in a given set of projection angles. The procedure is described in [1] (Section 4.3, p. 88), where it is used to generate a switching component in the context of lattice sets. For each new direction, a negated copy of the switching component is added, that is translated in the given direction. As shown in the Fig. 2, the same procedure applies directly to pixel images, and can also be used to create switching components that are constant, and non-zero, on a given region. The figure shows the subsequent addition of switching components in the horizontal, vertical, and diagonal direction, respectively.

Fig. 2. Example of a switching component for three directions, which is constant and nonzero on each of the black and white regions

Proposition 1. *Let $D = \{\theta_1, \dots, \theta_d\}$ be a given set of projection angles. Let $G = \{g_1, \dots, g_k\}$, $A \subset \mathbb{R}^2$ and $g \in G$. Let $f \in \mathcal{G}$ such that $f(x, y) = g$ for all $(x, y) \in A$. Then for each grey level $\tilde{g} \in \mathbb{R}$, there is an image $\tilde{f} \in \mathcal{F}$ such that $\tilde{f}(x, y) = \tilde{g}$ for all $(x, y) \in A$ and $R(f)(t, \theta_i) = R(\tilde{f})(t, \theta_i)$ for $i = 1, \dots, d$, $t \in \mathbb{R}$. Moreover, there is such an image $\tilde{f}$ that has at most $3k$ grey levels.*

Proof. (sketch) By the construction depicted in Fig. 2, an image can be created that has a constant value of 1 on A , has constant zero projections in all given directions and only contains grey levels from $\{-1, 0, 1\}$. Let $\rho = \tilde{g} - g$. By adding a multiple of ρ times the switching component to f , an image $\tilde{f}$ that conforms to the proposition can be created. This image will have at most $3k$ grey levels, included in the set $\{g_1 - \rho, g_1, g_1 + \rho, g_2 - \rho, g_2, g_2 + \rho, \dots, g_k - \rho, g_k, g_k + \rho\}$. Note that some of these grey levels may be negative, even if the image f has only nonnegative grey levels.

2.2 Discretization

In practice, a projection is measured at a finite set of detector cells, each measuring the integral of the object density along a line. Let m denote the total number of measured detector values (for all angles) and let $\mathbf{p} \in \mathbb{R}^m$ denote the measured projection data. We now discretize the images $f \in \mathcal{F}$ as well, represented on a square grid of width w . Let $n = w^2$, the number of pixels in the image. We assume that the image is zero outside this rectangle. Let $\mathbf{v} \in \mathbb{R}^n$ denote the discretized image of the object. The Radon transform for a finite set of angles can now be modeled as a linear operator $\mathbf{W}$, called the *projection operator*, that maps the image $\mathbf{v}$ to the projection data $\mathbf{p}$:

$$\mathbf{W}\mathbf{v} = \mathbf{p}. \quad (1)$$

We represent $\mathbf{W}$ by an $m \times n$ matrix $\mathbf{W} = (w_{ij})$. The vector $\mathbf{p}$ is called the *forward projection* or *sinogram* of $\mathbf{v}$.

The notation introduced above allows us to define the key problem of this paper:

Problem 3. Let $\mathbf{v} \in \mathbb{R}^n$ be an unknown image and let $g \in \mathbb{R}$ be an unknown grey level. Suppose that $\mathbf{p} = \mathbf{W}\mathbf{v}$ is given, and that a set $A \subset \{1, \dots, n\}$ is given such that $v_i = g$ for all $i \in A$. Determine g .

2.3 Grey Level Penalty Function

Let $A \subset \{1, \dots, n\}$ and $g \in \mathbb{R}$, as in Problem 3. Reorder the pixels $\{1, \dots, n\}$ and the corresponding columns of $\mathbf{W}$, such that

$$\mathbf{W} = (\mathbf{W}_A \mathbf{W}_B), \quad (2)$$

where $\mathbf{W}_A$ contains the columns of $\mathbf{W}$ corresponding to the pixels in A . We then have

$$\mathbf{W}\mathbf{x} = (\mathbf{W}_A \mathbf{W}_B) \begin{pmatrix} \mathbf{g} \\ \mathbf{v}_B \end{pmatrix} = \mathbf{p}, \quad (3)$$

where $\mathbf{g}$ denotes a constant vector for which all entries are g . This leads to

$$\mathbf{W}_B \mathbf{v}_B = \mathbf{p} - \mathbf{W}_A \mathbf{g}, \quad (4)$$

which provides a necessary condition for a grey level estimate to be correct: Eq. (4) must have a solution. Clearly, this condition is not always sufficient, as emphasized by Prop. 1. Yet, if $|A|$ is a relatively large fraction of n , it can be expected that the condition is also sufficient, at least in many cases. Note that in practice, it may not be possible to solve Eq. (4) exactly, due to noise and discretization errors. Given A , one can measure the consistency of a grey level $\tilde{g}$ with the projection data $\mathbf{p}$ using the following *grey level penalty function*

$$P(\tilde{g}) = \min_{\mathbf{v}_B} \|\mathbf{p} - \mathbf{W}_A \tilde{\mathbf{g}} - \mathbf{W}_B \mathbf{v}_B\|, \quad (5)$$

where $\|\cdot\|$ denotes a certain norm.

To minimize the grey level penalty, the penalty function must typically be evaluated in several points. A range of iterative methods are available for solving the minimization problem in Eq. (5). In this paper, we use the SIRT algorithm; see [8,9]. Let $\mathbf{v}^{(0)} = \mathbf{0}$. For $q = 1, 2, \dots$, let $\mathbf{r}^{(q)} = \mathbf{p} - \mathbf{W}\mathbf{v}^{(q-1)}$. In each iteration q , the current reconstruction $\mathbf{v}^{(q-1)}$ is updated, yielding a new reconstruction $\mathbf{v}^{(q)}$, as follows:

$$v_j^{(q)} = v_j^{(q-1)} + \frac{1}{\sum_{i=1}^n w_{ij}} \sum_{i=1}^m \frac{w_{ij} r_i^{(q)}}{\sum_{j=1}^m w_{ij}}. \quad (6)$$

The SIRT algorithm converges to a solution of Eq. (5) where the norm to be minimized is a weighted sum of squares [9].

An important physical constraint in tomography problems, is that the attenuation coefficients of the scanned materials, corresponding to the grey levels in the reconstructed image, cannot be negative. Therefore, it seems that solving Eq. (5) for *nonnegative* $\mathbf{v}_B$ would yield a more powerful penalty function. However, this problem typically requires significantly more computation effort compared to the unconstrained variant (depending on the norm). As an alternative, we use a heuristic adaptation of the SIRT algorithm, where nonnegative entries of $\mathbf{v}_B$ are set to 0 after each iteration. This approach was already suggested in [8], and often results in more accurate reconstructions than the unconstrained version. We denote this version of SIRT by *SIRT-P*.

3 Estimation Approach

In this section, we propose the *Discrete Gray Level Selection* approach (DGLS) for estimating grey levels from projection data.

Fig. 3. Schematic overview of the Discrete Gray Level Selection procedure

Fig. 3 shows a schematic overview of the steps involved in estimating one or more grey levels. The first part of the procedure requires user interaction. Based on an initial reconstruction, obtained by a conventional (i.e., non-discrete) reconstruction algorithm, the user selects a region, the *user-selected part* (USP), for which it seems clear that it belongs to one of the grey levels. In this phase, a significant amount of implicit prior knowledge can be input by the user, such as the fact that the scanned object contains no holes, or knowledge about its topology. However, it can be difficult to select a proper region if the number of projections is small. The selected region should be as large as possible, but should

certainly not contain any pixels that correspond to a different grey level. If the original image contains several grey levels, a region must be selected for each grey level that needs to be estimated. The regions A_i , along with the projection data, now form the input of the estimation algorithm.

The proposed algorithm estimates a single grey level at a time. After selecting the USP, an optimization algorithm is used to minimize the grey level penalty function. To evaluate the penalty function for a given grey level g , the sinogram $\mathbf{W}_{Ag}$ of the USP is first subtracted from the complete sinogram $\mathbf{p}$, forming the right-hand side of Eq. (4). Subsequently, consistency of the remaining part of the image with the remaining projection data is determined by evaluating the penalty function (e.g., using SIRT or SIRT-P). Based on the value of the penalty function, the grey level estimate is updated iteratively, until the penalty function is minimized. Evaluating the grey level penalty function can be computationally expensive. To find the minimum of this function using a small number of evaluations, Brent's method is used, as described in Chapter 5 of [10]).

4 Experiments and Results

Simulation experiments were performed based on four 256×256 phantom images: two binary images (Fig. 4a and b), one image with 3 grey levels (Fig. 4c) and one with 7 grey levels (Fig. 4d). From each phantom image, a parallel beam sinogram was computed with a varying number of projections, angular range, or noise level. Then, the phantom image was reconstructed using both the SIRT and SIRT-P algorithm. Finally, the grey levels of the reconstructed phantom were estimated, either with a simple median value (MED) of the user-selected part or with the proposed DGLS approach based on the same USP, using either SIRT or SIRT-P to compute the grey level penalty. For all images, the grey level of the background was assumed to be zero and was therefore not estimated. In all experiments, the USP for each grey level was computed by iteratively applying a binary erosion operation on the phantom region for that grey level, until a certain fraction of the pixels was left. Although in practice, the USP will be selected by the user, based on the reconstruction, using the phantom ensures that the USP is not affected by other parameters of the experiment. For both grey level estimation procedures, the absolute difference with respect to the true

Fig. 4. Phantom images used for our simulation experiments

Fig. 5. SIRT reconstructions of phantom image 4(c)

Fig. 6. SIRT reconstructions of phantom image 4(b)

Fig. 7. A USP of Phantom image 4(a)

grey level was computed. In case multiple grey levels had to be estimated (such as for phantom image 4(c) and 4(d)), the absolute differences were summed. The following series of experiments were run, varying only one parameter at a time:

- The **number of projections** from which the image was reconstructed was varied from 15 up to 180. In this experiment, the projections were equiangularly selected in the interval $[0^\circ, 180^\circ]$. For each grey level, the USP contains 25% of the pixels for that grey level. Fig. 5(a) and 5(b) show reconstructions of phantom image 4(c) for 180 and 15 projection angles, respectively. The results, plotted in Fig. 8, show that DGLS for both SIRT and SIRT-P generally yields much more accurate estimations than the MED estimations. Only when there are very few projection angles (e.g. 15), a significant error is visible. In phantom image 4(c) and 4(d), DGLS based on the unconstrained SIRT penalty function shows a large error when a small number of projection angles is used. We believe that this is related to the nonuniqueness issues of Prop. 1. As we can conclude that the DGLS method is able to compute an accurate estimate based on 30 projection angles, only 30 projection angles were used in all subsequent experiments.
- The **angular range** of the projections was varied from 180° down to 100° , from which 30 equiangular projections were selected. For each grey level, the USP contains 25% of the pixels for that grey level. Fig. 5(c) shows that reducing the angular range has a degrading effect on reconstructions, leading to artifacts. Fig. 9 shows that although the estimation error in absolute value

is larger than in the previous experiment, the DGLS error is significantly lower than the error of the median value.

- The **noise level** of the projections was varied. In this experiment, Poisson noise was applied to the projection data, for a varying source intensity. By *number of counts per detector element*, we refer to the measured detector count when there is no object blocking the path from source to detector. The higher this count, the higher the Signal-to-Noise ratio. For the reconstruction, 30 projection angles were equally spaced in $[0^\circ, 180^\circ]$. For each grey level, the USP again contains 25% of the pixels for that grey level. The effect of poisson noise on a reconstruction is visible in Fig. 6. Fig. 10 shows that for the investigated noise range, the noise level does not have a major impact on the estimation error, for both the MED and DGLS estimates. This can be explained by the fact that only 30 projections are used, such that the reconstruction errors due to the small number of projections are much more significant than those related to the noise in the projection data.
- The **size of the USP**, as a percentage of the total region for each grey level, was varied by iteratively applying an erosion filter as discussed above. Fig. 7 shows the user-selected part for phantom 4(a), 20% the size of the original phantom. Fig. 11 shows that accurate grey level estimation is possible even if the USP is relatively small, down to 10% of the object size.

The results suggest that the DGLS method is robust with respect to the number of projection angles, the angular range, the level of noise and the size of the user-selected part. Moreover, DGLS typically yields a significantly more accurate estimate compared to computing the median value of the user-selected part. The results also demonstrate that more accurate estimation can be achieved by using the heuristic SIRT-P method as a scoring function instead of SIRT.

On our test PC, running at 2.4GHz, the running time of a single grey level estimation in all experiments but the first one was around 4 minutes. This is mostly attributed to the SIRT or SIRT-P algorithm that has to be performed every time while minimizing the grey level penalty function.

5 Discussion

The grey level estimation problem, when posed in its general form, does not guarantee a unique solution. The experimental results show that even for a moderately large number of 15 projections, additional constraints may be necessary to obtain an accurate estimate of the grey levels using our proposed approach. Several techniques can be used to improve the accuracy of the estimate:

- **Use a More Accurate Scoring Function.** The projection distance, as computed using SIRT), does not incorporate nonnegativity constraints. Adding a nonnegativity heuristic to the reconstruction algorithm seems to result in improved accuracy of the grey level estimate, but its theoretical properties are hard to verify. A scoring function that incorporates both minimization of the projection distance and nonnegativity constraints (based on the nonnegative least squares problem) could result in more accurate scoring, at the expense of long running times.

Fig. 8. Error of grey level estimation with varying numbers of projection angles

Fig. 9. Error of grey level estimation with varying ranges of projection angles

- **Simultaneous Estimation of Several Grey Levels.** In the approach presented in this paper, each of the grey levels is estimated independently. Simultaneous estimation of all grey levels, where the grey level in all user-selected

Fig. 10. Error of grey level estimation with different noise levels

Fig. 11. Error of grey level estimation with different sizes of the user-selected part

parts is required to be constant, would provide more constraints for the estimation problem. However, this also leads to a higher dimensional search space (one dimension for each grey level), leading to longer running times.

However, restricting the reconstruction outside the user-selected part to non-negative values still does not capture the full set of constraints in Problem 1. To incorporate the fact that the entire image should contain only a small, discrete set of grey levels, it seems necessary to actually attempt to *compute* such a discrete reconstruction for varying grey levels, and check which grey levels correspond to a consistent reconstruction.

6 Conclusions

In this paper, we presented the DGLS approach: a semi-automatic method for estimating the grey levels of an unknown image from its projections. Grey level estimation is a necessary step before applying discrete tomography algorithms, as these algorithms typically assume the set of admissible grey levels to be known a priori. In its general form, the grey level estimation problem does not guarantee a unique solution. To allow for reliable estimates, additional prior knowledge must be incorporated. In our semi-automatic approach, this prior knowledge is included by letting the user select a region that is known to correspond to a constant grey level, based on an initial non-discrete reconstruction.

The proposed algorithm, which minimizes a penalty function while varying the grey level of the user-selected part, was shown to yield more accurate grey level estimates compared to direct estimation based on a continuous reconstruction. In particular, when a heuristic is used to enforce positivity of the image reconstructed during the penalty computation, accurate estimates can be obtained even from a small number of projection images, or a small angular range. An important question that remains for future research, is to determine what accuracy is actually required for the estimation step to use discrete tomography effectively.

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