Patterns in Discretized Parabolas and Length Estimation

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Abstract. We estimate the frequency of the patterns in the discretization of parabolas, when the resolution tends to zero. We deduce that local estimators of length almost never converge to the length for the parabolas.

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1 Introduction

Length estimation is an important domain of Image Analysis (see [1] for a review). In this paper, we will consider the problem of estimating the length of a curve from its discretizations at different resolutions. In particular we are interested in the comportment of estimators when the resolution tends to zero. We also restrict our study to special estimators called “local estimators” which consist in considering *patterns* which are pieces of fixed length of the discretized curve. Local estimators simply consist to fix a weight to each pattern and summing these weights to obtain the estimation of length (See Fig. 4 for illustration). So, if we want to study the estimated length by local estimators when the resolution tends to zero, we have at first to study the number of occurrences of a pattern of the discretization of digital curves. In fact an asymptotic result about the occurrence number of patterns for discretized general curves looks to be a quite hard problem, because the discretization process is not a continuous process (the integer part function is not continuous), so the estimation of the occurrence number of patterns cannot be deduced from Mathematical Analysis arguments, but by Number Theory arguments. The two first authors of this paper have already made this study for segments in [2]. In this paper we continue this work by considering another class of curves, the parabolas.

The paper is organized as follows: Section 2 describes the notations used in this paper, Section 3 will be devoted to the study of the frequency of patterns in parabolas, and finally Section 4 will apply the results of this study to the local estimators of length of parabolas. In this paper, some proofs are only sketched, the complete proofs are in the preprint [3].

2 Notations

In this section we precise the notations that will be used in all the paper.

- For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) the integer k such that $k \leq x < k+1$ (resp. $k-1 < x \leq k$).
- The fractional part of x is denoted $\langle x \rangle$ and is defined by $x = \lfloor x \rfloor + \langle x \rangle$.
- For $A, B \in \mathbb{Z}$, the discrete interval $\{A, A+1, \dots, B-1, B\}$ is denoted $\llbracket A, B \rrbracket$.
- Let m be a positive integer. A pattern of size m is a function ω from $\llbracket 0, m \rrbracket$ to $\mathbb{Z}$ such that $\omega(0) = 0$ and $\omega(k+1) \in \{\omega(k), \omega(k) + 1\}$ (see Fig. 1). The set of patterns of size m is denoted $\mathcal{P}_m$.
- If X and Y are two real numbers such that $Y > 0$ then $X \bmod Y$ is the real number such that $0 \leq X \bmod Y < Y$ and $\frac{X - X \bmod Y}{Y} \in \mathbb{Z}$.
- For $r \in \mathbb{R}$ and $E \subset \mathbb{R}^2$, $rE = \{(rx, ry) | (x, y) \in E\}$.

Fig. 1. The 16 patterns of size $m = 4$. Only the encircled one is not a digital segment.

3 Frequency of Patterns in Discrete Parabolas

Let $a, b \in \mathbb{R}$ such that $a < b$ and a derivable function $g : [a, b] \rightarrow \mathbb{R}$ which satisfies $0 \leq g'(x) \leq 1$ for all $x \in [a, b]$.

In all the following, for any $r > 0$ we use the notations:

- $A_r = \lceil \frac{a}{r} \rceil$, $B_r = \lfloor \frac{b}{r} \rfloor$, $N_r = B_r - A_r + 1$,
- $\mathcal{C}_r^g = r\{(X, Y) \in \mathbb{Z}^2 \mid A_r \leq X \leq B_r \text{ and } Y = \lfloor \frac{g(rX)}{r} \rfloor\}$.

The set $\mathcal{C}_r^g$ is the “naive” discretization of the graph of g at resolution r , and N_r is its number of points.

Let m be a positive integer. The pattern at position $X \in \llbracket A_r, B_r - m \rrbracket$ of size m of $\mathcal{C}_r^g$, denoted $\omega_{X,r,m}^g$ is defined by:

$$\omega_{X,r,m}^g(k) = \lfloor \frac{g(r(X+k))}{r} \rfloor - \lfloor \frac{g(rX)}{r} \rfloor.$$

The frequency of a pattern ω of size m in $\mathcal{C}_r^g$ is defined by:

$$F_r^g(\omega) = \frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid \omega_{X,r,m}^g = \omega\}}{N_r - m}.$$

The aim of this section is the study of $F_r^g(\omega)$ for some functions g . For this we will approximate the curve by its tangents which will also be discretized, so we need some notions about digital straight lines:

For $u \in [0, 1]$ and $v \in [0, 1]$, let us denote $s_m^{u,v}$ the pattern of size m defined by:

$$s_m^{u,v}(k) = \lfloor uk + v \rfloor.$$

A pattern of this form is called a digital segment of size m .

For any pattern ω ,

$$PI(\omega) = \{(u, v) \in [0, 1]^2 \mid \lfloor uk + v \rfloor = \omega(k) \text{ for any } k \in \llbracket 0, m \rrbracket\},$$

$$\text{pinf}_u(\omega) = \inf\{v \mid (u, v) \in PI(\omega)\}, \quad (1)$$

$$\text{psup}_u(\omega) = \sup\{v \mid (u, v) \in PI(\omega)\}, \quad (2)$$

$$\begin{aligned} FL_u(\omega) &= 0 && \text{if } \{v \mid (u, v) \in PI(\omega)\} = \emptyset \\ &= \text{psup}_u(\omega) - \text{pinf}_u(\omega) && \text{otherwise.} \end{aligned}$$

$PI(\omega)$ is called the preimage of ω , it is nonempty if and only if ω is a digital segment. $FL_u(\omega)$ is intuitively the frequency of the pattern ω in the discretized straight lines of slope u . (See [2,4] for more details and [5] for the generalization to slopes of planes).

In all the paper, the considered curves are parabolas corresponding to the function $g(x) = \alpha x^2$. We distinguish two cases: the case α irrational and the case α rational. In Subsection 3.1, we will see that for α irrational, the frequency $F_r^g(\omega)$ converges, when r is rational and tends to zero, to a quantity which can be expressed by using the function $x \mapsto FL_{g'(x)}(\omega)$. In Subsection 3.2, we study the case α rational, but we do not succeed to prove a similar result as in the case α irrational. Nevertheless we obtain a weaker result (The Tangent Lemma).

3.1 Parabolas of Equation $y = \alpha x^2$ with α Irrational

In this subsection we consider curves $\mathcal{C}_r^g$ for g defined on $[a, b]$ by $g(x) = \alpha x^2$ with α irrational, and $0 \leq a < b \leq \frac{1}{2\alpha}$. This last hypothesis is needed to have $0 \leq g'(x) \leq 1$ for $x \in [a, b]$.

The main result of this subsection is the following:

Theorem 1. *If $g(x) = \alpha x^2$ with $\alpha \notin \mathbb{Q}$ then for any pattern ω we have*

$$F_r^g(\omega) \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) dx \quad (3)$$

The rest of this subsection is devoted to the proof of this Theorem.

The first needed lemma shows that the discretization of a curve and the discretization of its tangent are similar near the origin of the tangent and when the resolution tends to zero:

Lemma 1 (Tangent Lemma). *If g is defined on $[a, b]$ by $g(x) = \alpha x^2$ with α irrational, and $0 \leq a < b \leq \frac{1}{2\alpha}$ then*

$$\frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid \omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} 0.$$

Lemma 1 is illustrated by Fig. 2. In the last lemma we consider $r \in \mathbb{Q}$ because its proof needs the irrationality of αr .

Fig. 2. Comparison of the discretization of the curve and the discretization of its tangent : here we have $\omega_{X,r,m}^g = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ for $m = 9$ but not for $m = 10$

Before starting the proof of Lemma 1 we need one more notation and two other lemmas. For $X \in \llbracket A_r, B_r \rrbracket$, we define $P_{r,k}(X)$ by:

$$P_{r,k}(X) = \frac{g(rX)}{r} + g'(rX)k = \alpha r X^2 + 2\alpha k r X = \alpha r((X + k)^2 - k^2).$$

This definition is motivated by the following lemma:

Lemma 2. *If $r < \frac{1}{\alpha m^2}$, then for any X we have $\omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ if and only if for all $k \in \llbracket 0, m \rrbracket$ we have $\langle P_{r,k}(X) \rangle < 1 - \alpha r k^2$.*

Proof.

$$\begin{aligned} \omega_{X,r,m}^g(k) &= \left\lfloor \frac{g(r(X+k))}{r} \right\rfloor - \left\lfloor \frac{g(rX)}{r} \right\rfloor \\ &= \left\lfloor \frac{g(rX)}{r} + g'(rX)k + \alpha r k^2 \right\rfloor - \left\lfloor \frac{g(rX)}{r} \right\rfloor \end{aligned}$$

We know that $\lfloor u + v \rfloor = \lfloor u \rfloor + \lfloor v \rfloor$ iff $\langle u \rangle + \langle v \rangle < 1$.

So $\omega_{X,r,m}^g(k) = \left\lfloor \frac{g(rX)}{r} + g'(rX)k \right\rfloor - \left\lfloor \frac{g(rX)}{r} \right\rfloor$ iff $\langle \frac{g(rX)}{r} + g'(rX)k \rangle + \langle \alpha r k^2 \rangle < 1$.

With the hypothesis $r < \frac{1}{\alpha m^2}$ we have $\langle \alpha r k^2 \rangle = \alpha r k^2$. But:

$$\begin{aligned} \lfloor \frac{g(rX)}{r} + g'(rX)k \rfloor - \lfloor \frac{g(rX)}{r} \rfloor &= \lfloor \langle \frac{g(rX)}{r} \rangle + g'(rX)k \rfloor - \lfloor \langle \frac{g(rX)}{r} \rangle \rfloor \\ &= s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}(k) \end{aligned}$$

So $\omega_{X,r,m}^g = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ iff for all $k \in \llbracket 0, m \rrbracket$ we have $\langle P_{r,k}(X) \rangle < 1 - \alpha r k^2$. $\square$

Lemma 3. *Let I be an interval of $[0, 1]$. We have $T_{r,k}(I) \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} \mu(I)$ where*

$$T_{r,k}(I) = \frac{1}{N_r} \text{card}\{X \in \llbracket A_r, B_r \rrbracket \mid \langle P_{r,k}(X) \rangle \in I\}$$

and $\mu(I)$ is the usual length of I .

The proof of this lemma uses Weyl's argument, as in the proof of Theorem 1 of [2] or Appendix A.1 of [5], but extended to the quadratic case following the same ideas as in [6, p6-7]. It is given in Appendix A.1 of [3].

Proof of Lemma 1. We recall that $\omega_{X,r,m}^g = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ iff for all $k \in \llbracket 0, m \rrbracket$ we have $\langle P_{r,k}(X) \rangle < 1 - \alpha r k^2$ so:

$$\frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid \omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \leq \frac{N_r}{N_r - m} \max_{k=0}^m T_{r,k}(I_{r,k})$$

where $I_{r,k} = [1 - \alpha r k^2, 1)$, so it is sufficient to show that $T_{r,k}(I_{r,k}) \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} 0$ for

any $k \in \llbracket 0, m \rrbracket$. Let $\varepsilon > 0$, there exists $R_1 > 0$ such that for any $r < R_1$ we have $\alpha r k^2 < \frac{\varepsilon}{2}$. We know by Lemma 3 that there exists $R_2 > 0$ such that for any $r < R_2$ we have $T_{r,k}(I_{R_0,k}) \leq \mu(I_{R_0,k}) + \frac{\varepsilon}{2}$. If r is such that $r < \min(R_1, R_2)$, we have:

$$\begin{aligned} S_{r,k}(I_{r,k}) &\leq S_{r,k}(I_{R_1,k}) \\ &\leq \mu(I_{R_1,k}) + \frac{\varepsilon}{2} \\ &= \alpha r k^2 + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This finishes the proof of Lemma 1. $\square$

Sketch of Proof of Theorem 1. We present here the main ideas of the proof of Theorem 1. The detailed proof is in Appendix A.2 of [3].

By using Lemma 1 we know that $F_r^g(\omega)$ has the same limit as r tends to zero as:

$$G_r^g(\omega) = \frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle} = \omega\}}{N_r - m}.$$

But $s_m^{x,y} = \omega$ is equivalent to $(x, \langle y \rangle) \in PI(\omega)$ so:

$$G_r^g(\omega) = \frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in PI(\omega)\}}{N_r - m}$$

which has the same limit as $H_r(PI(\omega))$ where

$$H_r(E) = \frac{\text{card}\{X \in \llbracket A_r, B_r \rrbracket \mid (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in E\}}{B_r - A_r + 1}.$$

By applying Lemma 3 with $k = 0$ to the piece of the curve $y = g(x)$ restricted to the domain $g'^{-1}(\alpha_1) \leq x \leq g'^{-1}(\alpha_2)$, we can prove:

$$H_r([\alpha_1, \alpha_2] \times I) \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} \frac{g'^{-1}(\alpha_2) - g'^{-1}(\alpha_1)}{b - a} \mu(I)$$

So by approximating $PI(\omega)$ as the union of rectangles

$$\bigcup_{i=1}^n [y_{i-1}, y_i] \times [\text{pinf}_{y_i}(\omega), \text{psup}_{y_i}(\omega))$$

we approximate $H_r(PI(\omega))$ by:

$$\sum_{i=1}^n \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b - a} FL_{y_i}(\omega)$$

which is a Riemann sum for $\int_a^b FL_{g'(x)}(\omega) dx$. □

Corollary 1. *If $g(x) = \alpha x^2$ with $\alpha \notin \mathbb{Q}$, then for any pattern ω which is not a digital segment we have*

$$F_r^g(\omega) \xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} 0.$$

Numerical Application: We illustrate Theorem 1 with an example. Consider the curve $\mathcal{C}$ defined $y = g(x) = \frac{1}{\sqrt{2}}x^2$ for x between $a = 0$ and $b = \frac{1}{\sqrt{2}}$, and the pattern ω of size $m = 3$ defined by $(\omega(0), \omega(1), \omega(2), \omega(3)) = (0, 1, 2, 2)$. We will compute the limit of the frequency of ω when the resolution tends to zero.

First we can compute easily $FL_\alpha(\omega)$ because $\alpha \mapsto FL_\alpha(\omega)$ is a continuous function which is affine between two m -Farey numbers (see [4]). So we deduce:

$$\begin{aligned} FL_\alpha(\omega) &= 0 && \text{if } \alpha \in [0, \frac{1}{2}] \\ &= 2\alpha - 1 && \text{if } \alpha \in [\frac{1}{2}, \frac{2}{3}] \\ &= 1 - \alpha && \text{if } \alpha \in [\frac{2}{3}, 1] \end{aligned}$$

Theorem 1 proves that:

$$\begin{aligned}
 F_r^g(\omega) &\xrightarrow[r \in \mathbb{Q}]{r \rightarrow 0} \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) dx \\
 &= \sqrt{2} \int_0^{\frac{1}{\sqrt{2}}} FL_{\sqrt{2}x}(\omega) dx \\
 &= \sqrt{2} \left(\int_{\frac{1}{2\sqrt{2}}}^{\frac{2}{3\sqrt{2}}} (2\sqrt{2}x - 1) dx + \int_{\frac{2}{3\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1 - \sqrt{2}x) dx \right) = \frac{1}{12}
 \end{aligned}$$

3.2 Parabolas of Equation $y = \alpha x^2$ with α Rational

Now we are interested in the case where α is rational. Theorem 1 can be generalized to α rational and to the irrational resolutions r because only the irrationality of αr is used in the proof of this theorem.

On the contrary, in the case α rational and rational resolutions, the only result we will prove in this subsection is about the Tangent Lemma. Moreover, we must impose some restrictions about the resolution r and the interval $[a, b]$ of definition of the parabola: $r\alpha = \frac{1}{p}$ where p is a prime number ; $a = 0$ and $b = \frac{1}{2\alpha}$. In all this subsection we suppose that these conditions are satisfied. Actually, we do not succeed to prove the Tangent Lemma in the general case for α rational.

Let $P_{r,k}(X) = r\alpha((X+k)^2 - k^2)$. We will prove that for any interval $I \subset [0, 1]$

$$\lim_{\substack{r \rightarrow 0 \\ \frac{1}{r\alpha} \text{ is prime}}} \frac{\text{card}\{X \in \llbracket A_r, B_r - k \rrbracket \mid \langle P_{r,k}(x) \rangle \in I\}}{N_r - k} = \mu(I)$$

Definition 1. Let p be a prime number and a be an integer number, we define the Legendre symbol $(\frac{a}{p})$ of a relatively to p by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } a \\ 1 & \text{if } a \text{ there exists } t \in \mathbb{Z} \text{ such that } a \bmod p = t^2 \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Property 1. (Pólya-Vinogradov inequality [7], [8, Chap. 23]) Let p be a prime number and M and N two positive integers then

$$\left| \sum_{n=M}^{M+N} \left(\frac{n}{p}\right) \right| < \sqrt{p} \log(p)$$

Corollary 2. Let $J = \llbracket M, M + N \rrbracket$ be an integer interval. Then

$$\left| \frac{\text{card}(J)}{2} - \text{card}\{y \in I \mid (\frac{y}{p}) = 1\} \right| < \sqrt{p} \log(p).$$

Lemma 4. *Let α be a rational number and assume that $a = 0$ and $b = \frac{1}{2\alpha}$. Then, for any interval $I \subset [0, 1]$*

$$\lim_{r \rightarrow 0} \frac{\text{card}\{X \in \llbracket A_r, B_r - k \rrbracket \mid \langle P_{r,k}(X) \rangle \in I\}}{N_r - k} = \mu(I)$$

where the limit is taken on the r such that $r > 0$ and $r\alpha = \frac{1}{p}$ where p is a prime number.

Proof. The function $X \mapsto X^2 \bmod p$ from $\llbracket 1, \frac{p-1}{2} \rrbracket$ to $\{Y \mid (\frac{Y}{p}) = 1\}$ is a bijection.

Put $H_r = \frac{\text{card}\{X \in \llbracket A_r, B_r - k \rrbracket \mid \langle P_{r,k}(x) \rangle \in I\}}{N_r}$. Then

$$\begin{aligned} H_r &= \frac{\text{card}\{X \in \llbracket k, \lfloor \frac{p}{2} \rfloor \rrbracket \mid \langle \frac{X^2 - k^2}{p} \rangle \in I\}}{N_r - k} \\ &= \frac{2 \text{card}\{X \in \llbracket k, \frac{p-1}{2} \rrbracket \mid \langle \frac{X^2 \bmod p - k^2}{p} \rangle \in I\}}{p + 1 - 2k} \end{aligned} \quad (4)$$

Thus

$$\text{card}\{X \in \llbracket 1, \frac{p-1}{2} \rrbracket \mid X^2 \bmod p \in J\} = \text{card}\{Y \in J \mid (\frac{Y}{p}) = 1\} \text{ where } J = pI + k^2.$$

So $|\text{card}\{X \in \llbracket k, \frac{p-1}{2} \rrbracket \mid X^2 \bmod p \in J\} - \text{card}\{Y \in J \mid (\frac{Y}{p}) = 1\}| < k$.

By using Pólya-Vinogradov inequality we have:

$$\left| \frac{\text{card}(J)}{2} - \text{card}\{X \in \llbracket 1, \frac{p-1}{2} \rrbracket \mid X^2 \bmod p \in J\} \right| < \sqrt{p} \log(p)$$

So, $\left| \frac{\text{card}(J)}{2} - \text{card}\{X \in \llbracket k, \frac{p-1}{2} \rrbracket \mid X^2 \bmod p \in J\} \right| < \sqrt{p} \log(p) + k$.

By (4) we have

$$\begin{aligned} \left| \frac{2}{p+1-2k} \frac{\text{card}(J)}{2} - \frac{2}{p+1-2k} \text{card}\{X \in \llbracket k, \frac{p-1}{2} \rrbracket \mid X^2 \bmod p \in J\} \right| \\ < \frac{2\sqrt{p} \log(p) + 2k}{p+1-2k} \end{aligned}$$

$$\text{Thus, } \lim_{r \rightarrow 0} \frac{\text{card}\{X \in \llbracket A_r, B_r - k \rrbracket \mid \langle P_{r,k}(X) \rangle \in I\}}{N_r - k} = \mu(I)$$

□

Theorem 2 (Tangent Lemma for rational slopes and special rational resolutions). *For any rational $\alpha > 0$ and any a, b such that $0 \leq a < b \leq \frac{1}{2\alpha}$ we have*

$$\frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid \omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \xrightarrow[r\alpha \text{ is prime}]{r \rightarrow 0} 0$$

Proof. The case where $a = 0$, $b = \frac{1}{2\alpha}$ can be proved exactly in the same way as for Lemma 1 by using Lemma 4 instead of Lemma 3. Consider the general case $[a, b] \subset [0, \frac{1}{2\alpha}]$. Then we know that:

$$\begin{aligned} & \frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \mid \omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \\ & \leq \frac{\lfloor \frac{1}{2\alpha r} \rfloor + 1 - m}{N_r - m} \cdot \frac{\text{card}\{X \in \llbracket 0, \lfloor \frac{b}{r} \rfloor - m \rrbracket \mid \omega_{X,r,m}^g \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{\lfloor \frac{1}{2\alpha r} \rfloor + 1 - m} \\ & \xrightarrow[\frac{1}{r\alpha} \text{ is prime}]{r \rightarrow 0} \frac{\frac{1}{2\alpha}}{b - a} \cdot 0 = 0 \quad \text{because of the case } a = 0, b = \frac{1}{2\alpha}. \quad \square \end{aligned}$$

Unfortunately we do not successfully generalize Theorem 1 to rational α and some rational resolutions even if experimentally Theorem 1 seems to be true in all the cases.

4 Application to Local Estimators of Length

A local estimator is given by a weight function p from the set $\mathcal{P}_m$ of patterns of size m to $\mathbb{R}$. The estimated length of the curve $y = g(x)$ where $g : [a, b] \rightarrow \mathbb{R}$ at resolution r is given by:

$$l(p, g, r) = r \sum_{k=0}^{\lfloor \frac{B_r - m - A_r}{m} \rfloor} p(\omega_{X,r,m}^g(A_r + km)).$$

Theorem 3. *Let a, b such that $0 \leq a < b$. The length estimated by a local estimator of a parabola $y = \alpha x^2$, $x \in [a, b]$, ($\alpha \leq \frac{1}{2b}$) converges when the resolution r is rational and tends to zero, to the length of the parabola only for a finite number of irrational numbers α .*

Sketch of proof. For this proof we use the notation $[f(x)]_a^b = f(b) - f(a)$.

It is easy to see that:

$$l(p, g, r) - \frac{(b-a)}{m} \sum_{\omega \in \mathcal{P}_m} p(\omega) F_r'^g(\omega) \xrightarrow{r \rightarrow 0} 0$$

where

$$F_r'^g(\omega) = \frac{\text{card}\{X \in \llbracket A_r, B_r - m \rrbracket \cap (A_r + m\mathbb{Z}) \mid \omega_{X,r,m}^g = \omega\}}{\lfloor \frac{B_r - m - A_r}{m} \rfloor}.$$

Consider again the curve defined by $g(x) = \alpha x^2$ for $\alpha \notin \mathbb{Q}$. The proof of Theorem 1 can be extended to prove that:

$$F_r'^g(\omega) \xrightarrow[\substack{r \rightarrow 0 \\ r \in \mathbb{Q}}]{r \rightarrow 0} \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) dx.$$

Fig. 3. Estimation of length of a curve from its discretization for two different resolutions: a) $l(p, g, \frac{1}{2}) = \frac{1}{2}(2p(\omega_1) + 3p(\omega_2) + 1p(\omega_4)) = 6.7$, b) $l(p, g, \frac{1}{4}) = \frac{1}{4}(4p(\omega_1) + 5p(\omega_2) + 2p(\omega_3) + 2p(\omega_4)) = 7.1$

So,

$$l(p, g, r) \xrightarrow[r \in \mathbb{Q}, r \rightarrow 0]{\frac{1}{m} \sum_{\omega \in \mathcal{P}_m}} p(\omega) \int_a^b FL_{g'(x)}(\omega) dx. \quad (5)$$

We know that $x \mapsto FL_x(\omega)$ is piecewisely affine ([4]). From this property we deduce that we can partition the interval $[0, \frac{1}{2b}]$ in a finite number of intervals $(I_k)_{0 \leq k \leq n}$ such that $L_{est}(\alpha) = \lim_{r \rightarrow 0} l(p, g, r)$ is of the form $\frac{A}{\alpha} + B\alpha + C$ on each interval I_k . (See Appendix A.3 of [3] for the details).

Let $L_{real}(\alpha)$ be the length of the parabola $\{(x, \alpha x^2) \mid x \in [a, b]\}$. We have:

$$L_{real}(\alpha) = \int_a^b \sqrt{1 + (2\alpha x)^2} dx = \left[\frac{x\sqrt{1 + (2\alpha x)^2}}{2} + \frac{\arg \sinh(2\alpha x)}{4\alpha} \right]_a^b$$

Suppose that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational numbers α , then there exists an interval I_k of the previous partition of $[0, \frac{1}{2b}]$ such that

$L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational numbers $\alpha \in I_k$. On I_k we know that $L_{est}(\alpha)$ has the form $\frac{A}{\alpha} + B\alpha + C$. The functions $\alpha \mapsto \alpha L_{real}(\alpha)$ and $\alpha \mapsto \alpha(\frac{A}{\alpha} + B\alpha + C)$ are holomorphic in a open set of $\mathbb{C}$ containing $[0, \frac{1}{2b}]$ and are equal for an infinite number of $\alpha \in I_k \subset [0, \frac{1}{2b}]$. So by Theorem on the zeros of holomorphic functions [9, Cha. 10] they are equal on $[0, \frac{1}{2b}]$. So:

$$\alpha L_{real}(\alpha) = A + B\alpha^2 + C\alpha \quad \text{for all } \alpha \in [0, \frac{1}{2b}]$$

We have:

$$\begin{aligned} \frac{\partial(\alpha L_{real}(\alpha))}{\partial \alpha} &= b\sqrt{1 + (2\alpha b)^2} - a\sqrt{1 + (2\alpha a)^2} \\ &= b - a + 2(b^3 - a^3)\alpha^2 + o(\alpha^2) \quad \text{when } \alpha \rightarrow 0 \end{aligned}$$

But $\frac{\partial(A+B\alpha^2+C\alpha)}{\partial \alpha} = 2B\alpha + C$, so $2(b^3 - a^3) = 0$ which is impossible if $b > a$. So the hypothesis that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational α is absurd. $\square$

Corollary 3. *Let a, b such that $0 \leq a < b$. The length estimated by a local estimator of a parabola $y = \alpha x^2$, $x \in [a, b]$, does not converge when the resolution is rational and tends to zero, to the length of the curve for almost all $\alpha \in [0, \frac{1}{2b}]$.*

Numerical application: Again, we take the curve $y = g(x) = \frac{1}{\sqrt{2}}x^2$ for x between $a = 0$ and $b = \frac{1}{\sqrt{2}}$. Suppose that we consider the local estimator Chamfer 5-7-11 ([10]) with $m = 2$, $p(000) = 2$, $p(001) = p(011) = \frac{22}{10}$, $p(012) = \frac{28}{10}$. With Equation (5), we can prove that the estimation of the length of the parabola given by this local estimator converges to $L_{est} = \frac{23}{40}\sqrt{2} \approx 0.813172$, this limit is different from the length of the parabola which is $\frac{1}{2} + \frac{1}{4}\sqrt{2}\log(1+\sqrt{2}) \approx 0.811612$.

Fig. 4. Figure showing Convergence Speed of the Chamfer 5-7-11 to its limits for the parabola $y = \frac{1}{\sqrt{2}}x^2$, $0 \leq x \leq \frac{1}{\sqrt{2}}$

Moreover Figure 4 shows how the length given by the estimator converges to its limit when the resolution tends to zero. It seems on this example that $l(p, g, r) - L_{est} = O(r)$.

5 Conclusion and Perspectives

In this paper, we have proved some local properties of discretizations of parabolas: First we show that locally discretization of parabola and discretization of its tangent often coincide (Tangent Lemma: Lemmas 1 and Theorem 2). In particular, asymptotically, the local patterns of a discretized parabola are digital segments. From this, we also give an explicit formula for the limit of frequency of a pattern of a parabola when the resolution tends to zero (Theorem 1). This has the important consequence that we can know to what tend local estimators of length for the parabolas, moreover it can be proved that this limit is often different from the length of the curve.

This work mainly brings two perspectives:

- The extension of Formula (3), which gives the limit of the frequency of pattern when the resolution tends to zero, to more general curves, in particular to the curves $y = P(x)$ when P is a polynomial of degree greater than 2.
- The application of this work for recognition of curve by just looking at patterns. For example, if the frequencies of patterns of a curve does not satisfy Theorem 1 then it is not a parabola of equation $y = \alpha x^2$.

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