Minimal Offsets That Guarantee Maximal or Minimal Connectivity of Digital Curves in nD

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Abstract. In this paper we investigate an approach of constructing a digital curve by taking the integer points within an offset of a certain radius of a continuous curve. Our considerations apply to digitizations of arbitrary curves in arbitrary dimension n . As main theoretical results, we first show that if the offset radius is greater than or equal to $\sqrt{n}/2$, then the obtained digital curve features maximal connectivity. We also demonstrate that the radius value $\sqrt{n}/2$ is the minimal possible that always guarantees such a connectivity. Moreover, we prove that a radius length greater than or equal to $\sqrt{n-1}/2$ guarantees 0-connectivity, and that this is the minimal possible value with this property. Thus, we answer the question about the minimal offset size that guarantees maximal or minimal connectivity of an offset digital curve.

Keywords: Digital geometry, digital curve, digital object connectivity, curve offset.

1 Introduction

Digital curves appear to be basic primitives in digital geometry. They have been used in volume modeling as well as in certain algorithms for image analysis, in particular in designing discrete multigrid convergent estimators [1–3].

A basic approach to modeling curves and surfaces in computer aided geometric design is the one based on computing the *offset* of a curve/surface Γ – the locus of points at a given distance d from Γ (see, e.g., [4–6] and the bibliography therein). As a rule, such sort of considerations resort to results from algebraic geometry, such as Grobner basis computation (see, e.g., [7, 8]). Note, however, that, usually, when looking for a curve offset defined by a distance d , one does not really care about the properties of the set of integer points enclosed by the offset. Exceptions (in 2D) are provided, e.g., by some recent works that define digital conics [9, 10].

In view of the above, it would indeed be quite natural to define a digital curve by the set of integer points within the offset of a continuous curve. With

no doubt it would be useful to know more about the pros and cons of digital curves obtained by an offset approach. In fact, this is the idea of the definition of a 2D digital line, which consists of the integer points that belong to a strip determined by two parallel lines at an appropriate distance from each other, so that the obtained digital set is connected. Assuring connectivity of a digital curve is a fundamental issue, as it can be seen, e.g., in [9, 11–17]. Offset approach has also been used to define digital circles and spheres, whose connectivity properties have been studied, as well [18, 19].

In the present paper we provide a complete answer (from a digital geometry point of view) to the most fundamental theoretical questions concerning digital curves defined by offsets. As main theoretical results, we first show that, in dimension n , if the offset radius is greater than or equal to $\sqrt{n}/2$, then the obtained digital curve features maximal connectivity. We also demonstrate that the radius value $\sqrt{n}/2$ is the minimal possible that always guarantees such a connectivity. Moreover, we prove that a radius length greater than or equal to $\sqrt{n-1}/2$ guarantees 0-connectivity, and that this is the minimal possible value with this property. All results apply to digitizations of arbitrary curves in an arbitrary dimension. With these two results we answer the question about the minimal offset size that guarantees maximal or minimal connectivity of an offset digital curve.

The paper is organized as follows. In the rest of the present section we recall some basic notions of digital geometry to be used in the sequel. Section 2, contains some subsidiary lemmas and constructions. In Section 3 we present the main theoretical results. We conclude with some final remarks in Section 4.

Some Definitions and Notations

Throughout we conform to terminology used in [20] (see also [21, 22]).

Our considerations take place in the *grid cell model* which consists of the grid cells of $\mathbb{Z}^n$, together with the related topology. In this model, a regular orthogonal grid subdivides $\mathbb{R}^n$ into n -dimensional hypercubes (e.g., unit squares for $n = 2$ or unit cubes for $n = 3$, also considered as n -cells) defining a class $\mathbb{C}_n^{(n)}$. These are usually called *hypervoxels*, or *voxels*, for short. Let $\mathbb{C}_n^{(k)}$ be the class of all k -dimensional cells of n -dimensional hypercubes, for $0 \leq k \leq n$. The grid-cell space $\mathbb{C}_n$ is the union of all classes $\mathbb{C}_n^{(k)}$, for $0 \leq k \leq n$. The $(n-1)$ -cells, 1-cells, and 0-cells of a voxel are referred to as *facets*, *edges*, and *vertices*, respectively.

We say that two voxels v, v' are k -adjacent for $k = 0, 1, \dots, n-1$, if they share a k -cell.

An n -dimensional (n D) digital object S is a finite set of voxels. A k -path (where $0 \leq k \leq n-1$) in S is a sequence of voxels from S such that every two consecutive voxels of the path are k -adjacent. Two voxels of S are k -connected (in S) iff there is a k -path in S between them. A subset G of S is k -connected iff there is a k -path connecting any two voxels of G .

A maximal (by inclusion) k -connected subset of a digital object S is called a k -(connected) component of S . Components are non-empty and distinct k -components are disjoint.

Given a set $M \subseteq \mathbb{R}^n$, by $|M|$ we denote its cardinality, by $M_{\mathbb{Z}}$ its *Gauss digitization* $M \cap \mathbb{Z}^n$, and by $\text{conv}(M)$ its convex hull.

Throughout the paper, the Euclidean norm is assumed. For a hyperball in $\mathbb{R}^n$ we will also use the term an *n-ball*. An *n-ball* of radius r and center c will be denoted by $B(c, r)$, and the corresponding sphere by $S(c, r)$. By $\overline{pq}$ we denote the segment with end-points p and q .

By $\{0, 1\}^n$ we denote the vertices of the unit hypercube. These are the points $(x_1, x_2, \dots, x_n)$ where $x_i = 0$ or 1 for $1 \leq i \leq n$.

Finally, a *curve* $\gamma \in \mathbb{R}^n$ is a continuous mapping $\gamma : I \rightarrow \mathbb{R}^n$, where I is a closed interval of real numbers.

2 Subsidiary Lemmas and Constructions

2.1 Lemmas

The following two folklore facts will be used in the proofs that follow.

Lemma 1. *Any closed n -ball $B \subset \mathbb{R}^n$ with a radius greater than or equal to $\sqrt{n}/2$ contains at least one integer point.*

Follows from the fact that the longest diagonal in a grid hypercube of $\mathbb{Z}^n$ has length $\sqrt{n}$.

Lemma 2. *Let A and B be digital sets, each of which k -connected. If there are points $p \in A$ and $q \in B$ that are k -adjacent, then $A \cup B$ is k -connected.*

Corollary 1. *If A and B are digital sets, each of which is k -connected, and $A \cap B \neq \emptyset$, then $A \cup B$ is k -connected.*

Lemma 3. *If a closed n -ball $B \subset \mathbb{R}^n$ contains exactly two integer points, then these are $(n - 1)$ -adjacent.*

Proof. Assume that B contains integer points p and q that are not $(n - 1)$ -adjacent. We will show that then B contains at least a third point.

Let $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n)$. Since p and q are not $(n - 1)$ -adjacent, at least two of their coordinates differ. W.l.o.g., let $p_n < q_n$ (so, we also have $p_i \neq q_i$ for at least one i , $1 \leq i \leq n - 1$).

Let B have center $c = (c_1, c_2, \dots, c_n)$ and radius r . Then $(c_1 - p_1)^2 + (c_2 - p_2)^2 + \dots + (c_n - p_n)^2 \leq r^2$ and $(c_1 - q_1)^2 + (c_2 - q_2)^2 + \dots + (c_n - q_n)^2 \leq r^2$. Now we observe that: if $(c_n - p_n)^2 < (c_n - q_n)^2$ (which is equivalent to $p_n + q_n > 2c_n$), then $q' = (q_1, q_2, \dots, q_{n-1}, p_n) \in B$; if $(c_n - p_n)^2 > (c_n - q_n)^2$ (i.e., iff $p_n + q_n < 2c_n$), then $p' = (p_1, p_2, \dots, p_{n-1}, q_n) \in B$; and if $(c_n - p_n)^2 = (c_n - q_n)^2$ (i.e., iff $p_n + q_n = 2c_n$), then both $p', q' \in B$. $\square$

We will also use the following fundamental lemma.

Lemma 4. *Given a closed n -ball $B \subset \mathbb{R}^n$ with $B_{\mathbb{Z}} \neq \emptyset$, $B_{\mathbb{Z}}$ is $(n - 1)$ -connected.*

Proof. We prove the statement by induction on $|B_{\mathbb{Z}}|$. If $|B_{\mathbb{Z}}| = 0$ or 1 , we are done. The case $|B_{\mathbb{Z}}| = 2$ is possible only if the two elements of $B_{\mathbb{Z}}$ are $(n-1)$ -adjacent (otherwise $B_{\mathbb{Z}}$ would contain a third element adjacent to one of the two, as follows from Lemma 3). Assume the statement is true for $|B_{\mathbb{Z}}| = k$, $k \geq 2$. Now let B be an n -ball with $|B_{\mathbb{Z}}| = k+1$. Let B^0 be an n -ball that contains exactly two points on its bounding sphere S^0 and $B_{\mathbb{Z}}^0 = B_{\mathbb{Z}}$.

Such a ball can easily be constructed, e.g., by performing the following steps. If S (the bounding sphere of B) contains exactly two integer points, we are done and set $S^0 = S$. If S contains more than two integer points, go to Step 3. If S contains exactly one integer point, go to Step 2. If S contains no integer points, then do the following. *Step 1.* Decrease the radius of B with a sufficiently small $\varepsilon > 0$, until the corresponding sphere S includes integer point(s). If there are exactly two such points, then we are done and set $S^0 = S$. If there is exactly one such point, go to Step 2, otherwise go to Step 3. *Step 2.* Let p be the integer point on S and O the center of S . Decrease the radius of S with a sufficiently small $\varepsilon > 0$ by moving its center along the line $\overline{Op}$ towards p until other integer point(s) meet S . If exactly one integer point meets S , we are done and set $S^0 = S$. Otherwise, go to Step 3. *Step 3.* Let p and q be the end-points of an edge of $\text{conv}(B_{\mathbb{Z}})$ and t the midpoint of $\overline{pq}$. Increase the radius of the corresponding bounding sphere S with $\varepsilon > 0$ by moving the center of S along $\overline{tO}$ in direction opposite to t , so that the obtained sphere S^0 contains p and q and encloses the same integer points as S , but no other integer points. Since the set of integer points is discrete, this is obviously possible for sufficiently small ε . Then the only integer points on the bounding sphere S^0 of the obtained ball B^0 will be p and q .

Let p and q be the only integer points on S^0 . Then, similar to the construction of S^0 , one can find a ball B^1 , such that $B_{\mathbb{Z}}^1 = B_{\mathbb{Z}}^0 \setminus p = B_{\mathbb{Z}} \setminus p$ (e.g., by decreasing the radius of B with a sufficiently small $\varepsilon > 0$ by moving its center along $\overline{Oq}$ towards q , while keeping q on S). The latter set is at least $(n-1)$ -connected by the inductive hypothesis. Analogously, we obtain that there is a ball B^2 , for which $B_{\mathbb{Z}}^2 = B_{\mathbb{Z}}^0 \setminus q = B_{\mathbb{Z}} \setminus q$, the latter set being $(n-1)$ -connected by the inductive hypothesis. Then, by Corollary 1, $B_{\mathbb{Z}}$ is $(n-1)$ -connected, as well. $\square$

We complete this section with two more technical facts for future reference.

Lemma 5. *Let $S(c, r)$ be a sphere that contains the points $(0, 0, \dots, 0)$, $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$. Then $c = (1/2, 1/2, \dots, 1/2)$ and $r = \sqrt{n}/2$ (and so, S contains all vertices of the unit hypercube, that are $(x_1, x_2, \dots, x_n)$, $x_i = 0$ or 1 for $1 \leq i \leq n$).*

Proof. Since $(0, 0, \dots, 0) \in S$, we have $c_1^2 + c_2^2 + \dots + c_n^2 = r^2$. Since $(1, 0, \dots, 0) \in S$, we have $(c_1 - 1)^2 + c_2^2 + \dots + c_n^2 = r^2$. Then $c_1^2 = (c_1 - 1)^2$, that yields $c_1 = 1/2$. Analogously, we get $c_2 = c_3 = \dots = c_n = 1/2$ and thus $r^2 = n/4$, so $r = \sqrt{n}/2$. Obviously, every point of $\{0, 1\}^n$ satisfies the sphere equation. $\square$

Lemma 6. *The n -ball $B(c + \varepsilon, \sqrt{n}/2)$, where $c = (1/2, 1/2, \dots, 1/2)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, and $|\varepsilon_i| < 1/2$, $\varepsilon_i \neq 0$ for at least one i , $1 \leq i \leq n$, contains at least one of the elements of $\{0, 1\}^n$ in its interior.*

Proof. W.l.o.g., assume that $\varepsilon_i \geq 0$ for $1 \leq i \leq k$ and $\varepsilon_i \leq 0$ for $k + 1 \leq i \leq n$, for some k , $0 \leq k \leq n$ (for $k = 0$ all ε_i 's are nonpositive while for $k = n$ all of them are nonnegative). $B(c + \varepsilon, \sqrt{n}/2)$ is defined by the inequality $(1/2 + \varepsilon_1 - x_1)^2 + (1/2 + \varepsilon_2 - x_2)^2 + \dots + (1/2 + \varepsilon_n - x_n)^2 \leq n/4$. The terms of the lemma also imply that

$$(1/2 + \varepsilon_1 - 1)^2 + \dots + (1/2 + \varepsilon_k - 1)^2 + (1/2 + \varepsilon_{k+1})^2 + \dots + (1/2 + \varepsilon_n)^2 < n/4, \quad (1)$$

since each additive does not exceed $1/4$ (equalities are possible if the corresponding $\varepsilon_i = 0$). Moreover, inequality (1) is strict, since at least one $\varepsilon_i \neq 0$. Hence, the point $(1, 1, \dots, 1, 0, 0, \dots, 0)$ belongs to the interior of B , as stated. $\square$

Next, we provide a dynamic geometric interpretation of the concept of offset. It will be instrumental to the proof of the main results which are presented afterward.

2.2 Geometric Interpretation

Let γ be a curve in $\mathbb{R}^n$ with endpoints p and q . Place an n -ball B of radius r at point p . The space included in the n -ball is part of the offset.

Now move B so that its center continuously runs over γ until it reaches the other endpoint q . See Fig. 1, left. The space traced by B while its center is moving from p to q constitutes the r -offset of γ . It will be denoted by $Off(\gamma, r)$. We will identify ball's position by the position of its center.

Fig. 1. Geometric interpretation. *Left:* The curve offset as a trace of a moving ball along the curve. *Middle:* An integer point t on the offset border is met by the moving ball. *Right:* An integer point t of the offset interior, when it is met by the moving ball, and when it leaves the ball.

When B is in its initial position (centered at p), it may or may not contain integer points. If the radius of B is sufficiently large, B always contains one or more integer points, as follows from Lemma 1.

During the motion of B along the curve, some integer points may leave it while others may enter it. Note that, depending on the curve specificity, it may be possible for an integer point to leave the ball, and later on to be met by it again. The following is a plain fact.

Fact 1. *During its motion from its initial position p to its final position q , the ball B meets all integer points that are contained in $\text{Off}(\gamma, r)$ and constitute the offset digitization of γ .*

The subregion of $\text{Off}(\gamma, r)$ traced by B when its center reaches a point $s \in \gamma$, $s \neq q$, will be called a *partial offset of γ at point s* . The set of integer points in such a partial offset will be called a *partial offset digitization of γ at s* .

3 Main Results

In this section we resolve the problem about the offset sizes that guarantee certain types of connectivity.

3.1 Minimal Offset Preserving Maximal Connectivity: First Main Theorem

It was shown in [23] that if $D(L)$ is a digital straight line defined by a cylindrical r -offset of a straight line in $\mathbb{R}^3$, then $D(L)$ is at least 1-connected provided that $r \geq \sqrt{3}$. The following theorem strengthens this last result in several ways. First, it applies to arbitrary curves, in arbitrary dimension, and proves maximal $(n-1)$ -connectivity. Moreover, the required value of the radius is twice smaller than the one used in [23]. Finally, it is the minimal possible value that guarantees $(n-1)$ -connectivity. Thus, in fact, this theorem closes the considered issue.

Theorem 1. *Let γ be a curve in $\mathbb{R}^n$, $n \geq 2$. Let $D(\gamma, r)$ be the set of integer points within an r -offset $\text{Off}(\gamma, r)$. If $r \geq \sqrt{n}/2$, then $D(\gamma, r)$ is $(n-1)$ -connected.*

Moreover, the value $r = \sqrt{n}/2$ is the minimal possible that guarantees $(n-1)$ -connectivity of $D(\gamma, r)$.

Proof. We will prove that for $r = \sqrt{n}/2$, $D(\gamma, r)$ is $(n-1)$ -connected. This will obviously imply the validity of the first part of the theorem for any $r \geq \sqrt{n}/2$.

We proceed by induction on the number of points in a partial offset digitization. Let p and q be the two endpoints of γ . Place an n -ball B of radius $\sqrt{n}/2$ at a point p . Let M_p be the partial offset digitization of γ at p (i.e., $M_p = B(p, r)_{\mathbb{Z}}$). By Lemma 1, $|M_p| \geq 1$. If we choose an arbitrary element of $B(p, r)_{\mathbb{Z}}$, it is $(n-1)$ -connected. Note that by Lemma 4, $B(p, r)_{\mathbb{Z}}$ is $(n-1)$ -connected, as well.

Let M_s be the partial offset digitization of γ at an arbitrary point $s \in \gamma$. By Lemma 4, $M_s \geq 1$. As an inductional hypothesis, suppose that M_s is $(n-1)$ -connected. If $M_s = D(\gamma, r)$, we are done. Otherwise, move B along γ towards q .

During the motion, at a certain moment B will meet integer point(s) that are not currently in it. Note however that these may already be in the partial offset digitization. If this is the case, keep moving until reaching integer point(s) not yet included in $D(\gamma, r)$. Then these points are added to the current content of $D(\gamma, r)$. Assume that this happens when the center of B is at some point $c \in \gamma$. We will show that M_c is $(n-1)$ -connected.

Let t be a new integer point met by B at position c , and $t \notin M_s$. Let $B(c, r)$ be the ball in that position. To simplify the calculations, assume without loss of generality that the center $c = (c_1, c_2, \dots, c_n)$ belongs to the positive ortant and that t is at the origin of the coordinate system (i.e., $t = (0, \dots, 0)$). Then the sphere $S(c, r)$ that bounds $B(c, r)$ has an equation

$$(c_1 - x_1)^2 + (c_2 - x_2)^2 + \dots + (c_n - x_n)^2 = (\sqrt{n}/2)^2 = n/4, \quad (2)$$

and the distance from c to t equals the radius of the sphere, that is

$$\sqrt{c_1^2 + c_2^2 + \dots + c_n^2} = \sqrt{n}/2. \quad (3)$$

Consider all $(n-1)$ -neighbors of t in the nonnegative ortant, that are the points

$$(0, 0, \dots, 0, 1), (0, 0, \dots, 1, 0), \dots, (1, 0, \dots, 0, 0). \quad (4)$$

If some of these was (were) already in the partial offset when B was approaching t , we are done. Assume the opposite, i.e., none of these was in the partial offset before that moment. This is possible only if at the point when B meets t , the points from (4) are either outside or on $S(c, r)$.

Point $(1, 0, \dots, 0)$ does not belong to the interior of $B(c, r)$ if and only if

$$(c_1 - 1)^2 + c_2^2 + \dots + c_n^2 \geq n/4.$$

Keeping Eq. (2) in mind, the latter is equivalent to $c_1 \leq 1/2$. Analogously, we obtain

$$c_1 \leq 1/2, c_2 \leq 1/2, \dots, c_n \leq 1/2. \quad (5)$$

The latter inequalities imply

$$c_1^2 + c_2^2 + \dots + c_n^2 \leq n/4. \quad (6)$$

If at the meeting point of B with t some of the points (4) is outside $S(c, r)$, then the corresponding inequality in (5) would be strict. Then inequality (6) would be strict too, which contradicts Eq. (3).

Thus, it remains to consider the possibility when all $(n-1)$ -neighbors of t are met by ball B at the same time (and thus all of them at the same time become elements of M_c). Lemma 5 implies that then all points $\{0, 1\}^n$ will belong to $S(c, r)$. By Lemma 6, there is a point $w \in \{0, 1\}^n$ that has been in the interior of B as B 's center was approaching c , i.e., $w \in M_s$. Now we have that both M_s and $B(c, r)_{\mathbb{Z}} = \{0, 1\}^n$ are $(n-1)$ -connected, and $w \in M_s \cap B(c, r)_{\mathbb{Z}}$. Then, $M_c = M_s \cup B(c, r)_{\mathbb{Z}}$ is $(n-1)$ -connected, as well, by Corollary 1.

Fig. 2. The integer points within a straight line offset of radius $\sqrt{2}/2 - \varepsilon$ may not be 1-connected, even for an arbitrarily small $\varepsilon > 0$

Since the set $D(\gamma, r)$ is finite and in view of Fact 1, eventually all integer points within the offset will enter $D(\gamma, r)$, which will be $(n - 1)$ -connected. This completes the proof of the first part of the theorem.

Fig. 2 provides an example, which demonstrates that the value $r = \sqrt{n}/2$ is the minimal possible that guarantees $(n - 1)$ -connectivity of the digital line defined by an r -offset. This completes the proof of the theorem. $\square$

Remark 1. The example of Fig. 2 makes also clear that Theorem 1 fails to hold if the cylindrical offset is not a closed set containing the whole cylindrical boundary.

Theorem 1 suggests that $D(\gamma, \sqrt{n}/2)$ can be considered as an $(n - 1)$ -connected digital curve that is a digitization of a given continuous curve γ . Note that every point of that digitization is at a distance from γ no greater than $\sqrt{n}/2$.

3.2 Minimal Offset Preserving Minimal Connectivity: Second Main Theorem

In this section we obtain a lower bound for an offset radius r that guarantees 0-connectivity of $D(\gamma, r)$. Moreover, we show that this radius value is the minimal possible with this property.

Theorem 2. *Let γ be a curve in $\mathbb{R}^n$, $n \geq 2$. Let $D(\gamma, r)$ be the set of integer points within an r -offset $\text{Off}(\gamma, r)$. If $r \geq \sqrt{n-1}/2$, then $D(\gamma, r)$ is at least 0-connected.*

Moreover, the value $r = \sqrt{n-1}/2$ is the minimal possible that guarantees 0-connectivity of the digital line defined by r -offset.

Proof. We will prove that for $r = \sqrt{n-1}/2$, $D(\gamma, r)$ is 0-connected. This will clearly imply the validity of the first part of the theorem for any $r \geq \sqrt{n-1}/2$.

Fig. 3. *Left:* If the digitized curve is too short, its r -offset (where $r < \sqrt{n}/2$) may not contain any integer points. *Right:* A curve that fully crosses a pixel.

If $r < \sqrt{n}/2$, then obviously the curve offset may contain no integer points (see Fig. 3, left). If that is the case, i.e., if $D(\gamma, \sqrt{n-1}/2) = \emptyset$, we are done. In the rest of the proof we will assume the nontrivial case $D(\gamma, \sqrt{n-1}/2) \neq \emptyset$.

With a reference to the *Geometric Interpretation* from Section 2.2, consider a curve $\gamma \subset \mathbb{R}^n$ with endpoints p and q . Place an n -ball B of radius r at point p .

To prove the first part of the theorem, we proceed by induction on the number of $(n-1)$ -facets of hypercubes intersected by γ . (Note that two consecutive intersections may occur at the same $(n-1)$ -facet.) We exclude from further consideration non-interesting cases such as the one in Fig. 3, left, assuming that γ goes through (i.e., “enters” and “leaves”) at least one grid hypercube (Fig. 3, right).

Let Q be a grid hypercube intersected by γ . It has $2n$ $(n-1)$ -facets $Q_1^{(n-1)}, Q_2^{(n-1)}, \dots, Q_{2n}^{(n-1)}$. Let $Q_i^{(n-1)}$ be one of these facets intersected by γ and let $u = \gamma \cap Q_i^{(n-1)}$. Denote by $L_i^{(n-1)}$ the affine hull of $Q_i^{(n-1)}$. It is a hyperplane in $\mathbb{R}^n$ that is a translated copy of a Euclidean subspace $\mathbb{R}^{n-1}$. Clearly, $L_i^{(n-1)}$ contains a subset of the grid-points $\mathbb{Z}^n$ which is a translated copy of a $\mathbb{Z}^{n-1}$ space. Denote it by $\mathbb{Z}_{L_i}^{n-1}$. Let $B_i^n(u, \sqrt{n-1}/2)$ be an n -ball centered at u . Then $B_i^{n-1}(u, \sqrt{n-1}/2) = B_i^n(u, \sqrt{n-1}/2) \cap L_i^{(n-1)}$ is an $(n-1)$ -ball with the same center and radius as B_i^n . By Lemma 1 applied to $B_i^{n-1}(u, \sqrt{n-1}/2)$ and $L_i^{(n-1)}$, we have that $B_i^{n-1}(u, \sqrt{n-1}/2)$ contains an integer point from $\mathbb{Z}_{L_i}^{n-1}$.

We have already seen that before the first intersection, the partial offset digitization may be an empty set. If it is non-empty, then the partial offset will contain integer points that are vertices of the same hypercube (the one which contains the portion of the curve that determines the considered partial offset). Hence, the partial offset digitization will be at least 0-connected.

Assume that after k intersections, the partial offset digitization $S_k \subseteq D(\gamma, \sqrt{n-1}/2)$ of integer points within the partial $\sqrt{n-1}/2$ -offset of γ traversed so far, is 0-connected. The last set of integer points added to the current partial offset digitization includes the integer points in an $(n-1)$ -ball

Fig. 4. A 2D example of a 0-connected digital curve obtained by an offset of radius $\sqrt{n-1}/2 = 1/2$. The points of the digital curve are marked by small circles.

Fig. 5. *Top:* A 3D example of a 0-connected digital curve obtained by an offset of radius $\sqrt{n-1}/2 = \sqrt{2}/2$. The points of the digital curve are marked by small circles. *Bottom:* Offset sections at levels $z = 0$ (left), $z = 1$ (middle), and $z = 2$ (right).

$B_k^{n-1}(u, \sqrt{n-1}/2)$, which in turn includes the integer points on an $(n-1)$ -facet $Q_k^{(n-1)}$ of some grid hypercube Q . (Note: Here k labels the last $(n-1)$ -facet intersected by γ ; these k facets may belong to different grid hypercubes, in general).

Moving along γ inside Q towards crossing another $(n-1)$ -facet of Q , the n -ball B may involve different sets of integer points. By Lemma 4, each of them is $(n-1)$ -connected. Moreover, every time when the ball contains some integer points, some of these will be among the cube vertices. The reason is that the integer point that is closest to the ball center will obviously belong to the n -cube that contains the ball center. Hence, all new integer points that are met by the ball B will be 0-connected to S_k .

Eventually, the ball center will meet an $(n-1)$ -facet $Q_{k+1}^{(n-1)}$ of Q at a certain point $v = \gamma \cap Q_{k+1}^{(n-1)}$. At that moment, $D(\gamma, \sqrt{n-1}/2)$ will include some vertices of $Q_{k+1}^{(n-1)}$, which follows from Lemma 1. However, each of these vertices is at least 0-adjacent to those of $Q_k^{(n-1)}$. Thus the set of all integer points included

Fig. 6. *Top:* A 3D example of a disconnected digital set obtained by an offset of radius slightly less than $\sqrt{n-1}/2 = \sqrt{2}/2$. The points of the set are marked by small circles. *Bottom:* Offset sections at levels $z = 0$ (left), $z = 1$ (middle), and $z = 2$ (right).

in $D(\gamma, \sqrt{n-1}/2)$ when the center of B reaches point v , will be at least 0-connected. See Figures 4 and 5 for two- and three-dimensional examples, respectively.

The validity of the second part follows from the example presented in Fig. 6. $\square$

Theorem 2 suggests to consider $D(\gamma, \sqrt{n-1}/2)$ as a 0-connected digital curve which is a digitization of a curve γ . Every point of that digitization is at a distance from γ no greater than $\sqrt{n-1}/2$.

4 Concluding Remarks

In this paper we investigated an approach of constructing a digital curve by taking the integer points within an offset of a certain radius of the curve. We showed that if the offset radius is greater than or equal to $\sqrt{n}/2$, then the obtained digital curve features maximal connectivity. We also demonstrated that the radius value $\sqrt{n}/2$ is the minimal possible that always guarantees such a connectivity. Moreover, we showed that radius length $\sqrt{n-1}/2$ guarantees 0-connectivity, and that this is the minimal possible value with this property. All results apply to digitizations of arbitrary curves in arbitrary dimensions.

The presented theoretical results may imply definitions of special classes of digital curves. Thus, one can obtain analytical definitions of $(n-1)$ -connected and 0-connected digital straight lines in arbitrary dimensions. Such definitions have been obtained by the authors. There is evidence that digital straight lines (or curves, in general) would be “thick,” especially in higher dimensions. However, we have experimentally tested that three-dimensional digital lines of these types have quite satisfactory appearance when projected on the computer screen. Because of space restrictions, detailed description is postponed to the full length journal version of the paper.

Acknowledgements

The authors thank the three anonymous referees for their critical remarks and useful suggestions.

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