On the Blurring of the Funk–Radon Transform in Q–Ball Imaging

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Abstract. One known issue in Q–Ball imaging is the blurring in the radial integral defining the Orientation Distribution Function of fiber bundles, due to the computation of the Funk–Radon Transform (FRT). Three novel techniques to overcome this problem are presented, all of them based upon different assumptions about the behavior of the attenuation signal outside the sphere densely sampled from HARDI data sets. A systematic study with synthetic data has been carried out to show that the FRT blurring is not as important as the error introduced by some unrealistic assumptions, and only one of the three techniques (the one with the less restrictive assumption) improves the accuracy of Q–Balls.

1 Introduction

High Angular Resolution Diffusion Imaging (HARDI) allows the characterization of complex tissue microarchitectures beyond one single fiber bundle per image voxel. Therefore it has become a very interesting topic in the recent literature [1,2,3,4]. Among the existing techniques, Q–Balls [5,6] have gained especial interest [7,8] for being fast and easy to estimate [9], and not needing further assumptions on the behavior of the diffusion signal outside the sampled sphere.

This technique is based on the integration of the attenuation signal in the equators of the sphere, estimating the Orientation Distribution Function (ODF) as the radial projection of the probability density along the corresponding axis. This is the so-called Funk–Radon Transform (FRT), whose main problem is that it is only an approximation of the radial integral defining the ODF. The error in the estimation of this integral produce the angular blurring of the ODF [6]. On the other hand, a recent study [8] has shown that the Diffusion Orientation Transform (DOT), as introduced in [3], may outperform Q–Balls in some situations, even when it is based upon the unrealistic assumption that the attenuation signal shows a mono-exponential decay. Based on this result, we propose three novel techniques to overcome FRT blurring from assumptions related to the one in [3]. They are tested with a systematical methodology similar to [8]. As a result we conclude, **first**, that the error (blurring) due to the FRT has less impact than the error introduced by the aforementioned assumption on the attenuation signal. **Secondly**, since such an assumption produces very accurate results with the DOT using very similar numeric schemes, the problem with Q–Balls relies on the estimation of the ODF instead of any other orientation information.

2 Theory

2.1 Characterization of Water Diffusion in the White Matter

Under the assumption of narrow pulses, the probability density for the displacement of water molecules to a position $\mathbf{R}$ for one single fiber bundle is related to the attenuation signal by the Stejskal–Tanner equation [10]:

$$P(\mathbf{R}) = \frac{1}{\sqrt{(4\pi^2\tau)^3|\mathcal{D}|}} \exp\left(\frac{-\mathbf{R}^T\mathcal{D}^{-1}\mathbf{R}}{4\tau}\right) \Leftrightarrow E(\mathbf{q}) = \exp(-b\mathbf{g}^T\mathcal{D}\mathbf{g}), \quad (1)$$

where $\mathbf{q} = q\mathbf{g}$, $\|\mathbf{g}\| = 1$, $b = 4\pi^2\tau q^2$ is the magnitude of the sensitizing gradients, τ is the effective diffusion time and the positive-definite matrix $\mathcal{D}$ is the diffusion tensor. For complex micro-architectures the Gaussian model in eq. (1) no longer holds and $P(\mathbf{R})$ can be computed in terms of the Fourier transform of $E(\mathbf{q})$ [11]:

$$P(\mathbf{R}) = \mathfrak{F}\{E(\mathbf{q})\}(\mathbf{R}) = \iiint_{\mathbb{R}^3} E(\mathbf{q}) \exp(-2\pi i \mathbf{q}^T \mathbf{R}) d\mathbf{q}, \quad (2)$$

where the expression of $E(\mathbf{q})$ in eq. (1) has to be substituted by:

$$E(\mathbf{q}) = \exp(-4\pi^2\tau q^2 D(q, \mathbf{g})) < 1, \quad (3)$$

where $D(q, \mathbf{g})$ is a positive function, the Apparent Diffusion Coefficient (ADC), defined for each spatial direction $\mathbf{g}$. In general the ADC depends on q , but for the tensor model $D(q, \mathbf{g}) = \mathbf{g}^T \mathcal{D} \mathbf{g}$ and this is not the case; the diffusion process may be characterized then by the sampling of the attenuation signal $E(\mathbf{q})$ in a sphere of a given radius q_0 , $E(q_0\mathbf{g})$. The DOT (see [3]) relies on the over-simplified assumption that the ADC does not depend on q , $D(q, \mathbf{g}) = D(\mathbf{g})$.

2.2 The Orientation Distribution Function

Although the sampling of $E(q_0\mathbf{g})$ for a given q_0 does not completely characterize the diffusion process, it is often enough to infer not the detailed behavior of $P(\mathbf{R})$, but only its underlying orientation information, associated to the presence of fiber bundles in these same directions. The ODF is defined in [5,6] as:

$$\Psi(\mathbf{r}) \equiv \Psi(\theta, \phi) = \int_0^\infty P(R\mathbf{r}) dR = \frac{1}{2} \int_{-\infty}^\infty P(R\mathbf{r}) dR, \quad (4)$$

where $R = \|\mathbf{R}\|$ and $\mathbf{r} = [\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta]^T$. Although the ODF is not a true probability density (see [5]) it provides useful orientation information, as has been widely reported [7,8]. From eqs. (2) and (4) it follows:

$$\begin{aligned} 2\Psi(\mathbf{r}) &= \int_{-\infty}^\infty \iiint_{\mathbb{R}^3} E(\mathbf{q}) e^{-2\pi i \mathbf{q}^T \mathbf{r}} d\mathbf{q} dR = \iiint_{\mathbb{R}^3} E(\mathbf{q}) \int_{-\infty}^\infty e^{-2\pi i \mathbf{q}^T \mathbf{r}} dR d\mathbf{q} \\ &= \iiint_{\mathbb{R}^3} E(\mathbf{q}) \delta(\mathbf{q}^T \mathbf{r}) d\mathbf{q} = \iint_{\langle \mathbf{r} \rangle^\perp} E(\mathbf{s}) d\mathbf{s}, \end{aligned} \quad (5)$$

where $\langle \mathbf{r} \rangle^\perp$ is the orthogonal set to the span of $\mathbf{r}$, i.e., the ODF at direction $\mathbf{r}$ may be computed as the integral of $E(\mathbf{q})$ in the plane perpendicular to $\mathbf{r}$.

2.3 The Funk–Radon Transform

For a given $\mathbf{r}$ the computation of the ODF $\Psi(\mathbf{r})$ requires to integrate $E(\mathbf{q})$ in the plane perpendicular to $\mathbf{r}$, but HARDI techniques allow only to characterize $E(\mathbf{q})$ in a circumference of radius q_0 inside this plane. The principle of Q–Ball imaging is to reduce the integral in eq. (5) to the integral in the circumference $S_{\mathbf{r}} \equiv \{\mathbf{q} | \mathbf{q}^T \mathbf{r} = 0, \|\mathbf{q}\| = q_0\} \subset \langle \mathbf{r} \rangle^\perp$ to compute the FRT of $E(q_0 \mathbf{g})$ as [5]:

$$\begin{aligned} \mathcal{G}\{E(q_0 \mathbf{g})\}(\mathbf{r}) &\triangleq \oint_{S_{\mathbf{r}}} E(q_0 \mathbf{g}) q_0 d\mathbf{g} \propto \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} P(\rho, \varphi, R) J_0(2\pi q_0 \rho) \rho d\varphi d\rho dR \\ &\simeq \int_{-\infty}^{\infty} P(R\mathbf{r}) dR = 2\Psi(\mathbf{r}), \end{aligned} \quad (6)$$

where (ρ, φ, R) are the coordinates of a cylindrical system with the z axis aligned with $\mathbf{r}$: the FRT is proportional to the integral of $P(\mathbf{R})$ not along $\mathbf{r}$ but inside a tube along $\mathbf{r}$ which has the shape of a Bessel function J_0 . Q–Balls obviate the need to characterize the whole $E(\mathbf{q})$ (they take into account its value only for q_0) at the expense of blurring the radial integral of $P(\mathbf{R})$. It is commonly assumed that this is the main drawback of Q–Ball imaging.

2.4 Beyond the Funk–Radon Transform

Given the nice results of DOT when assuming that the ADC is constant with q [3], even better than Q–Balls in certain situations [8], our aim is to use this same assumption to reduce the blurring due to the computation of the FRT.

Artificial Increase of the b -Value. Increasing the value of q_0 reduces the width of the Bessel kernel J_0 and therefore the blurring [5]. Assuming that $D(q, \mathbf{g}) \simeq D(q', \mathbf{g})$ for similar values of q and from eq. (3):

$$\begin{aligned} E(q'_0 \mathbf{g}) &= \exp\left(-4\pi^2 \tau q_0'^2 D(q'_0, \mathbf{g})\right) \\ &\simeq \exp\left(-4\pi^2 \tau q_0'^2 D(q_0, \mathbf{g})\right) = E(q_0 \mathbf{g})^{q_0'^2/q_0^2} = E(q_0 \mathbf{g})^\xi, \end{aligned} \quad (7)$$

and so we will refer to **Q–Balls– ξ** as the FRT of $E(q_0 \mathbf{g})^\xi$, $\xi > 1$.

Integration in the Whole Orthogonal Plane to $\mathbf{r}$. If we now assume that $D(q, \mathbf{g}) \simeq D(\mathbf{g})$ for all q , the integral in eq. (5) may be explicitly computed. Using the auxiliar cylindrical coordinates (ρ, φ, R) introduced above:

$$\begin{aligned} 2\Psi(\mathbf{r}) &= \int_0^{2\pi} \int_0^{\infty} E(\rho, \varphi, 0) \rho d\rho d\varphi \simeq \int_0^{2\pi} \int_0^{\infty} \exp(-4\pi^2 \tau \rho^2 D(\mathbf{g}(\varphi))) \rho d\rho d\varphi \\ &= \int_0^{2\pi} \frac{1}{8\pi^2 \tau D(\mathbf{g}(\varphi))} d\varphi \Rightarrow \Psi(\mathbf{r}) \simeq \mathcal{G}\left\{\frac{1}{16\pi^2 \tau} D^{-1}(q_0 \mathbf{g})\right\}(\mathbf{r}), \end{aligned} \quad (8)$$

so we will refer to **Q–Balls–ADC** as the FRT of the inverse of the ADC.

Application of Stokes' Theorem. Instead of the integral in all the plain orthogonal to $\mathbf{r}$, we may compute the integral in the circle Ω inside $S_{\mathbf{r}}$ ($S_{\mathbf{r}} \equiv \partial\Omega$); since $E(\mathbf{q})$ shows in general an exponential decay, the error committed in this way will be small. Using once again the cylindrical coordinates system with $\mathbf{e}_z \equiv \mathbf{r}$, this approximation may be identified with the flux integral:

$$\begin{aligned} 2\Psi(\mathbf{r}) &\simeq \iint_{\Omega} E(\rho, \varphi, 0) dA = \iint_{\Omega} E(\rho, \varphi, 0) \mathbf{e}_z \cdot d\mathbf{A} \\ &= \oint_{S_{\mathbf{r}}} F_{\varphi}(q_0, \varphi, 0) \mathbf{e}_{\varphi} d\mathbf{l} = \int_0^{2\pi} F_{\varphi}(q_0, \varphi, 0) q_0 d\varphi, \end{aligned} \quad (9)$$

for some $\mathbf{F}(\rho, \varphi, z) = F_{\varphi}(\rho, \varphi, z) \mathbf{e}_{\varphi}$ such that $\nabla \times \mathbf{F} = E_{\rho} \mathbf{e}_{\rho} + E_{\varphi} \mathbf{e}_{\varphi} + E \mathbf{e}_z$, by virtue of Stokes' theorem (note that E_{ρ} and E_{φ} are irrelevant since they do not contribute to the flux integral). Given the expression of the curl in cylindrical coordinates, F_{φ} and E must be related in the way:

$$\begin{aligned} \frac{1}{\rho} \frac{\partial \rho F_{\varphi}(\rho, \varphi, z)}{\partial \rho} &= \exp(-4\pi^2 \tau \rho^2 D(\rho, \varphi, z)) \simeq \exp(-4\pi^2 \tau \rho^2 D(q_0, \varphi, z)) \\ \Rightarrow F_{\varphi}(\rho, \varphi, z) &\simeq \frac{-\exp(-4\pi^2 \tau \rho^2 D(q_0, \varphi, z))}{8\pi^2 \tau \rho D(q_0, \varphi, z)} + \frac{\Theta(\varphi, z)}{\rho}, \end{aligned} \quad (10)$$

where $\Theta(\varphi, z)$ is a constant with respect to ρ : $\mathbf{F}$ needs to be non-singular at $\rho = 0$ for Stokes' theorem to apply, so we must choose $\Theta(\varphi, z) = (8\pi^2 \tau D(\varphi, z))^{-1}$. This is only a mathematical artifact, and has no other meaning in terms of the assumptions made. From eqs. (9) and (10) we define **Q-Balls-Stokes** as:

$$2\Psi(\mathbf{r}) \simeq \int_0^{2\pi} q_0 F_{\varphi}(q_0, \varphi, 0) d\varphi \simeq \int_0^{2\pi} \frac{1 - E(q_0, \varphi, 0)}{8\pi^2 \tau D(q_0, \varphi, 0)} d\varphi \propto \mathcal{G} \left\{ \frac{1 - E(q_0 \mathbf{g})}{D(q_0 \mathbf{g})} \right\}(\mathbf{r}). \quad (11)$$

Note that in eq. (10) we need to assume that the ADC is constant only in a local sense (in a differential environment of q_0), since we only need to define $\mathbf{F}$ for q_0 . It is worth to compare this with Q-Balls- ξ , where D should be constant in a range of values of q (the larger ξ , the wider the range) and with Q-Balls-ADC, where the ADC should be constant for all q . We may highly relax the assumption of constant ADC at the expense of neglecting the integral outside Ω .

3 Methods

3.1 Practical Computation of the Estimators

The three proposed estimators based on Q-Balls (Q-Balls- ξ , Q-Balls-ADC and Q-Balls-Stokes) may be computed as the FRT of a given 3-D function (see above); therefore, we use in all cases the method and parameters suggested in [9] to find the Spherical Harmonics (SH) expansion of the corresponding signal and then to analitically compute its FRT. Note that we have obviated all the constants relating the FRT with the estimators; this is not an issue since, as suggested in [5], we normalize the resulting ODF so that their minimum is 0 and they sum to 1. For the DOT, we use the parametric implementation in [3] with $R_0 = 12\mu\text{m}$ (a larger R_0 makes the DOT too sensitive to noise).

3.2 Assessment of the Accuracy Resolving Fiber Crossings

A DWI signal has been generated from a synthetic architecture comprising two crossing fiber directions in a known angle. The error between local maxima of the ODF and the ground–truth directions has been measured whenever the two fibers have been correctly detected. This methodology has been extensively used in the related literature [1,2,3,4,5,6,8,9]. In [3,8] the diffusion signal is generated with a model based on isotropic diffusion inside a bounded cylinder; however, we prefer to use the more standard methodology of multi–tensor approaches (a linear combination of eq. (1)) for three reasons:

1. The multi–tensor approach has been extensively validated, see [1,2,4,5,6,9].
2. According to our experience, both approaches perform very similar.
3. The bounded cylinder is a simplified model for the diffusion inside a neural axon, which typically has a diameter of about $5\mu\text{m}$. Comparing this to the voxel size (1–2 mm.), it is easy to appreciate that one single voxel describes not the microscopic diffusion inside a nervous cell but the macroscopic behavior given by several hundreds or more nervous fibers. This yields a mixture of independent and (roughly) identically distributed bounded cylinder statistics, and therefore a mixture of Gaussians is a more correct model.

3.3 Setting Up of the Experiments

We use a similar methodology to that of [8]. For several combinations of b values ($b = 1500, 2500$ and 3500mm/s) and gradient directions ($N_g = 61$ and 121) we measure the angular error committed in the estimation of the local maxima of two fiber bundles (matched to realistic biological parameters). We use order $L = 6$ in the SH expansion in all cases. For the noisy scenarios, we corrupt the diffusion signal with Rician noise adding a complex Gaussian noise with standard deviation σ and computing its envelope. Noise power is parameterized by the Peak Signal to Noise Ratio (PSNR), defined as the quotient between the value of the baseline and σ . Results are an average for 100 Montecarlo trials, and in this case we consider that a given estimator is able to find the two fibers if they are detected in more than the 50% of the trials.

4 Results and Discussion

In Fig. 1 the results without noise contamination are shown. **First**, note that the results are consistent with those previously reported in [8]: increasing the b value or the number of gradient directions improves the capability detection for all estimators, and the angular error is decreased as well. For a given configuration, Q–Balls perform worse than DOT, which is the same as saying that Q–Balls need a greater b value or more gradient directions to achieve an accuracy similar to DOT [8]. **Second**, if we compare traditional Q–Balls with the estimators based on a constant (at least in a range of b values) ADC (i.e., Q–Balls– ξ and Q–Balls–ADC), regular Q–Balls perform better in all cases: although assuming a constant

Fig. 1. Angular error in the recovering of two fiber bundles vs. the original angle between their directions, for the six configurations tested and for all estimators. The diffusion signals have not been contaminated with Rician noise.

Fig. 2. 3-D plot of the orientation functions (ODF for Q-Balls based estimators and $P(R_0\mathbf{r})$ for DOT), for $b = 3500$, $N_g = 121$ and an angle of 60° . Red axis represent local maxima of the estimators, and green axis the ground-truth directions.

ADC allows to reduce (for Q-Balls- ξ) or completely eliminate (for Q-Balls-ADC) the blurring due to the FRT, the error introduced by this oversimplified assumption does not compensate its benefit. Note that Q-Balls-ADC perform worse than Q-Balls- ξ except for very high b -values (when they perform very similar), so the more restrictive the assumption the worse the accuracy. **Third**, Q-Balls-Stokes perform better than regular Q-Balls in all cases (and even better than DOT for $b = 1500$ and 121 gradient directions), although the improvement

Fig. 3. Angular error vs. the inverse of the PSNR, for $N_g = 121$, and different b values and crossing angles, for all estimators

is quite subtle; in this case, the constant ADC assumption is needed only in a local (differential) sense, which is far more realistic. For Q–Balls–Stokes, the error due to the assumption has to be added to the error due to the integration inside $S_{\mathbf{r}}$ and not the whole orthogonal plane to $\mathbf{r}$, but they still improve the performance of Q–Balls. For illustrative purposes, we show in Fig. 2 a 3–D plot of the orientation functions given by each estimator: DOT is able to yield well defined lobes, meanwhile Q–Balls based estimators produce wider lobes and therefore a higher uncertainty in the location of fiber directions. Q–Balls–Stokes yield a very similar ODF to Q–Balls, but for Q–Balls– ξ and Q–Balls–ADC the lobes are even more blurred; once again, we may **conclude** that the constant ADC assumption introduces a high error in the estimation of the ODF which does not compensate the reduction of the FRT blurring, unless the assumption is applied in a local sense.

To test the behavior in the presence of noise, we vary the noise power σ and study the angular error (see Fig. 3). **First**, note that in general regular Q–Balls are still better than Q–Balls– ξ and Q–Balls–ADC, and Q–Balls–Stokes better than Q–Balls. **Second**, Q–Balls and Q–Balls–Stokes are more stable to noise than DOT: its accuracy is worsened in the presence of noise, but not as much as DOT accuracy does. **Third**, note that Q–Balls and Q–Balls–Stokes may be preferable to DOT for very noisy scenarios and large angles of crossing or low b values; this issue has been previously reported for regular Q–Balls in [8].

5 Conclusion

One of the benefits of Q–Ball estimation of fiber populations is that it does not require to make any assumption on the behavior of the diffusion signal;

this advantage carries out the drawback of the blurring in the radial integral defining the ODF, driving to broadened lobes in the orientation information. This drawback may be palliated by including some sort of assumptions. However, we have shown that the mono-exponential decay model in general introduces an important error, more important than the blurring inherent to Q-Balls. The same model has been successfully used in the DOT, which has been shown to be more accurate than Q-Balls in general. Since the implementations used here for both of them use the same numerical scheme (based on the SH expansion of the attenuation signal, compare [9] and [3]), the reason for this difference has to be in the orientation function Q-Balls estimate, which is, the ODF: the ODF is more sensible than the probability profile $P(R_0\mathbf{r})$ computed with the DOT. On the other hand, we have introduced a new technique, Q-Balls-Stokes, to improve the accuracy of regular Q-Balls; although its benefit is quite subtle, it outperforms Q-Balls in all cases, and even may outperform, for noisy scenarios with low b values (or high crossing angles), the DOT.

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