Forcing and Type Theory

Thierry Coquand

Department of Computer Science and Engineering,
Göteborg University and Chalmers University of Technology, Sweden
`coquand@chalmers.se`

We present the technique of *forcing* from a constructive point of view. The connections with proof theory [1] and constructive mathematics are not so surprising given that Cohen’s original discovery of forcing was “motivated by an attempt to prove analysis consistent” and the idea that statements which seemed to involve infinities “could be reduced to pieces of finite informations” [9]. The interest of combining forcing and realizability was pointed out in the proof of Goodman’s Theorem [5].

We explain first how forcing allows to explain non effective objects (such as non principal ultrafilters, or well-ordering of the reals) in term of classical logic and dependent choices [7]. The combination of classical logic and dependent choices have a computational interpretation [3,4,6]. However, the associated computational behaviour is difficult to describe. We show then how to describe in type theory the addition of one Cohen real. We make explicit the computations involved in such an extension. One application, similar to the ones described in the reference [2], is the uniform continuity of functionals on Cantor space definable in type theory. By iterating this extension, we get a computational interpretation of universal quantification over Cantor space.

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