Continuous Time Markov Chain Model of Asset Prices Distribution

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Abstract. The aim of this paper is to introduce a new model of a financial asset prices distribution. It is known that the probability distribution of an asset prices or returns is unknown in reality. The general model of asset prices based on continuous time Markov chains is proposed. For this reason the interarrivals between two price states are approximated by mixture of exponential distributions. Numerical-analytic approach is used to obtain the probability distribution of asset prices. The developed software allows creating the space of an asset prices, the matrix of transition rates among states, a system of equations to find the steady state probabilities of price states and solves the system of equations by method of imbedded Markov chains.

Keywords: Asset prices distribution, a mixture of exponential distributions, continuous time Markov chain, and numerical-analytic model.

1 Introduction

It should be noted that, in practice, we do not observe asset prices following continuous-variable, continuous-time processes. For example, stock prices are restricted to discrete values and changes can be observed only when exchange is open. Nevertheless, the discrete-variable and continuous-time process proves to be useful model for stock prices.

To price and hedge derivative securities, it is crucial to have a good model of the probability distribution of the underlying product. The most famous continuous time model is the celebrated Black - Scholes model [1] and discrete one is classical Cox-Rubinstein model [2], which uses the normal distribution to fit the log returns of the underlying asset. One of the main problems with these models is that the data suggest that the log returns of stocks are not normally distributed. So other more flexible distributions are needed. Some authors suggest the underlying normal distribution to replace by a more sophisticated one. Examples of such, which can take into account skewness, excess kurtosis and other features, are the Variance Gamma [3], the Normal Inverse Gaussian [4], the CGMY (named after Carr, Geman, Madan and Yor) [5], the Hyperbolic Model [6] and the Meixner [7] distribution. Including such features makes analytic modelling less tractable, and potentially makes numerical modelling a more attractive alternative. In the following sections the algorithm of modeling asset

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prices by Markov chains is proposed. Armed with the model of price dynamics, an investor can:

- Calculate theoretical prices for derivative securities
- Measure the amount of risk associated with holding risky securities.

2 Markov Property of Stock Prices

The dynamics of asset prices are reflected by uncertain movements of their values over time. Some authors [8, 9] state that Efficient Market Hypothesis (EMH) is one possible reason for the random behaviour of the asset price. The EMH basically states that past history is fully reflected in present prices and markets respond immediately to new information about the asset.

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variables and the way that that the present has emerged from the past are irrelevant.

Stock prices are usually assumed to follow a Markov process. These processes are important models of security prices, because they are often realistic representation of true prices and yet the Markov property leads to simplified computations. If the stock price follows a Markov process, our predictions of the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is the price now. Predictions are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

If stock price process $S = \{S_t, 0 \le t \le T\}$ is Markovian and if we denote by $F = \{F_t, 0 \le t \le T\}$ the natural filtration of *S* (intuitively, F_t contains all market information up to time *t*), then we can write for a well-behaved function f*:* $E[f(S_{T)}| \; F_t] = E[f(S_T)|S_t]$. The stock price process takes values in some countable set *E*, called the state space. If $S_t = j \in E$, we shall say ”the process is in state *j* at time *t*”. The most common situation is for the state to be a scalar, but frequently it is more convenient for the state to be a vector.

3 Approximation of the Probability Distribution of Stock Prices

Our aim is to construct the stock price dynamics as a continuous time Markov chain with countable space of states. To find the space of states and transition rates between them we have to construct price movement distributions up and down for a given stock. To get Markov process the distribution of time length of stock price rising or decreasing must be exponential with parameter μ. Unfortunately, usually it is insufficient, and then a convenient representation for more general distributions is the Coxian formulation [8]. This formulation, by means of fictitious phases, allows the duration of generating stock price rate of transition up or down to be described by a linear combination of stochastic variables. Thus, generation of price movement is a continuous succession of *k* phases, each having exponential service time distribution of rate μ_j, $j = 1,2,\cdots,k$. After phase *j*, a stock price leaves the phases with probability ($1 - p_j$). The stock price can occupy only one phase at a time. Therefore, there can be at most one stock price within the set of phases at any time.

Let us consider a general probability distribution function *G*(*t*) of stock prices. Useful approximation of this function can be obtained by the mixture and convolutions of exponential (phase-type) distributions. Then a Markov chain with a countable space of states and continuous time can represent the evolution of stock price dynamics. Suppose we let $m_k, k = \overline{1,3}$ denotes the *k*th non-central moment, i.e. $E\ [X^k]$, where *X* is a random variable of price movement time. Construct a random variable *X*, which can be represented as:

$$X = \begin{cases} X_1 \text{ with prob. } p_2; \\ X_1 + X_2 \text{ with prob. } p_1 p_2; \\ \qquad \ldots \\ X_1 + X_2 + \ldots + X_2 \text{ with prob. } p_1^{n-1} p_2; \\ \qquad \ldots \end{cases}$$

where X_i, $i = 1,2,$ are independent random variables having exponential distributions with means $1/\mu_1$ and $1/\mu_2$ respectively; $p_1 + p_2 = 1$. The random variable *X* equals to the sum of independent variables with random number *N* of summands. *N* is non-negative, integer-valued random variable with $E(N) < \infty$ having geometrical distribution. Its probability density is the following

$$f(x) = p_2 \mu_1 \left(\frac{\mu_2 - \mu_1}{\mu_2 p_2 - \mu_1} e^{-\mu_1 x} - \frac{\mu_2 p_1}{\mu_2 p_2 - \mu_1} e^{-p_2 \mu_2 x} \right)$$

Moment matching is a common method for approximating distributions. Though two-moment approximations are common, they may lead to serious error when the coefficient of variation ν, (the standard deviation divided by the mean) is high. The first three moments of any non degenerate distribution with support on [0,∞) can be matched by the distribution (2).

To obtain the values of the parameters μ_1, μ_2, p_1 and p_2 of approximation, a complex system of non-linear equations needs to be solved:

$$\begin{cases} \dfrac{1! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^2} - \dfrac{\mu_2 p_1}{\mu_2^2 p_2^2} \right) = m_1; \\ \dfrac{2! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^3} - \dfrac{\mu_2 p_1}{\mu_2^3 p_2^3} \right) = m_2; \\ \dfrac{3! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^4} - \dfrac{\mu_2 p_1}{\mu_2^4 p_2^4} \right) = m_3; \\ p_1 + p_2 = 1. \end{cases}$$

The solution of the system is the following:

$$\mu_2 = \frac{g_2 - g_1^2}{g_1^3 - 2g_1 g_2 + g_3}, g_k = \frac{m_k}{k!}, k = \overline{1,3};$$

$$\mu_1 = \frac{1 + \mu_2 g_1 \pm \sqrt{(1 - \mu_2 g_1)^2 + 4\mu_2{}^2 \left(g_2 - g_1^2\right)}}{2g_1 - 2\mu_2 \left(g_2 - g_1^2\right)};$$

$$p_1 = \frac{\mu_2 (\mu_1 g_1 - 1)}{\mu_2 (\mu_1 g_1 - 1) + \mu_1}; \quad p_2 = \frac{\mu_1}{\mu_2 (\mu_1 g_1 - 1) + \mu_1}.$$

The exponential stages are shown graphically in Fig.1.

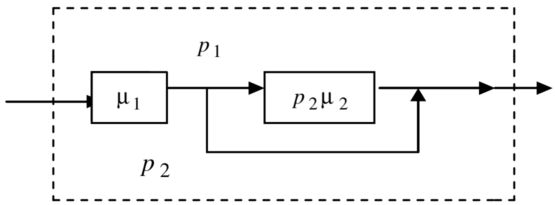

Fig 1. The diagram of two exponential phases

4 Calculation of Steady State Distribution of Markov Chain

In constructing the approximating Markov chain, three decisions need to be made. First, one must choose the set of discrete prices, i.e. $S = \{S_0, S_1, ..., S_n\}$. The second decision is generating the transition rates among states, and the third - calculating steady state distribution of asset prices.

In this section we will use so called the event language to generate the space of stock prices and transition matrix between them. Denote the time elapsed between two consecutive time observations by $\Delta Y = Y_i - Y_{i-1}, i = 1,2,...$

We approximate the distributions of price movement time up and down from the historical data of selected stock prices by the formulas respectively

$$f_u(y) = p_1^u \mu_1^u \left(\frac{\mu_2^u - \mu_1^u}{\mu_2^u p_2^u - \mu_1^u} e^{-\mu_1^u y} - \frac{\mu_2^u p_2^u}{\mu_2^u p_2^u - \mu_1^u} e^{-\mu_2^u p_2^u y} \right)$$

$$f_d(y) = p_1^d \mu_1^d \left(\frac{\mu_2^d - \mu_1^d}{\mu_2^d p_2^d - \mu_1^d} e^{-\mu_1^d y} - \frac{\mu_2^d p_2^d}{\mu_2^d p_2^d - \mu_1^d} e^{-\mu_2^d p_2^d y} \right)$$

A Markov chain with the countable space of states and continuous time can describe the dynamics of stock price movement. To construct a numerical model of the system, the approach proposed in will be applied.

The set of events in the system:

$$E = \{e_1^u, e_2^u, e_3^u, e_4^u, e_1^d, e_2^d, e_3^d, e_4^d\}$$

where

$e_1^{u(d)}$ – beginning of price movement up (down);

$e_2^{u(d)}$ – completed the stage of price movement up (down) with probability $p_1^{u(d)}$ in the first phase;

$e_3^{u(d)}$ – completed the stage of price movement up (down) with probability $p_2^{u(d)}$ in the first phase;

$e_4^{u(d)}$ – completed the stage of price movement up (down) in the second phase;

The set of transition rates:

$Intens = \{\mu_1^u, p_2^u\mu_2^u, \mu_1^d, p_2^d\mu_2^d\}$,

where

$\mu_1^{u(d)}$ - rate of price movement up (down) in the first phase;

$\mu_1^{u(d)}p_2^{u(d)}$ - rate of price movement up (down) in the second phase;

Let us consider an asset observed on a discrete time scale $\{0,1,\cdots,t,\cdots,T\}, T<\infty$ having S_t as market stock value at time *t*. To model the stochastic process $(S_t, t=0,1,\cdots,T)$ we suppose that the asset has known minimal and maximal values so that the set of all possible values is the closed interval $[S_{\min}, S_{\max}]$. For example, if S_0 is the value of the asset at time 0, we can put

$$S_0 = (S_{\max} + S_{\min})/2$$

$$S_k = S_0 + k\Delta,\ k = 1,\cdots,N$$

$$S_{-k} = S_0 - k\Delta,\ k = 1,\cdots.N$$

$$\Delta = (S_{\max} - S_{\min})/2N$$

N being chosen arbitrarily. This implies the total number of states is 2*n* + 1. In what follows, we order these states in the naturally increasing order and use the following notation for the state space:

$$I = \{-N,-(N-1),\cdots,0,1,\cdots,N\}.$$

We can also introduce different step lengths following movements up and down and so consider respectively Δ, Δ'. It is also possible to let $S_{\max} \to \infty$ and $T \to \infty$ particularly to get good approximation results.

To model the dynamics of stock prices we need know the phase in which is the stock price. The state space of the system is completely specified by the set of triples

$$B = \{(b_1, b_2, b_3)\},\ b_1 \in I,$$

where

$$b_2 = \begin{cases} 0, \text{if the stock price is not changing up;} \\ 1, \text{if the stock price is moving up in the first phase;} \\ 2, \text{if the stock price is moving up in the second phase.} \end{cases}$$

$$b_3 = \begin{cases} 0, \text{if the stock price is not changing down;} \\ 3, \text{if the stock price is moving down in the first phase;} \\ 4, \text{if the stock price is moving down in the second phase.} \end{cases}$$

The dynamics of stock price movement can be described in the event language. As an example, the description of the fourth event using pseudo code is represented bellow.

e_4^u :
```
      if  b2 = 2  and   b1 < n
        then  b1 ← b1 + 1; b2 ← 0
        Intense ← μ1^u p1
         end  then
    end if
end  e2^u
```

The software for automatic construction of numerical models is created [9]. The software consists of:

- The language of a model specification
- A program for automatic generation of all possible states (the set of stock prices) and transition rates among them
- A program for calculation of steady state probabilities of continuous Markov chain.

The states of Markov process are generated from an initial state. All possible transitions from this state are considered. When this step is completed, the current state is marked, and one of the newly obtained states becomes the current state. The generation process terminates when all the states in the list have been marked and no new state is obtained. Let $\Lambda_N = \left(\lambda_{ij}^{(N)}\right)_{N \times N}$ be the matrix of transition rates for a Markov chain on the set of states $S = \{S_0, \ldots, S_N\}$. Then

$$\lambda_{ij}^{(k)} = \lambda_{ij}^{(k+1)} + \frac{\lambda_{i,k+1}^{(k+1)}\,\lambda_{k+1,j}^{(k+1)}}{\overline{S}_{k+1}^{(k+1)}}, \quad i, j = \overline{0, k}; \quad k = \overline{N-1, 1};$$

$$\overline{S}_{k+1}^{(k+1)} = \sum_{\substack{j=0 \\ j \neq k+1}}^{N} \lambda_{k+1,j}^{(k+1)};$$

$$r_1^{(1)} = 1; \; r_i^{(k+1)} = \begin{cases} r_i^{(k)}, \quad i = \overline{0, k}; \\ \dfrac{\sum_{j=1}^{N-1} r_j^{(k)} \lambda_{ji}^{(k+1)}}{\overline{S}_i^{(k+1)}}, i = k+1, k = \overline{1, N-1}; \end{cases}$$

$$q_i = \frac{r_i^{(N)}}{\sum_{i=1}^{N} r_j^{(N)}}, \quad i = \overline{0, N}.$$

The probabilities of stock price states are calculated by the following formula:

$$p(k) = \sum_{b_2, b_3} q(k, b_2, b_3), \; k = -N, \cdots, N \,,$$

where $q(k, b_2, b_3)$ is the probability of the price at the fixed phase.

5 Numerical Example

This section discusses the actual implementation of the model's methodology discussed in the previous sections.

Data used in the analyses is the *Microsoft Corporation* daily log-returns which start from February 2007 and end in May 2007. The statistical analysis showed that the skewness is negative and the kurtosis is higher than three, it can be said that the *Microsoft Corporation* log-returns in the sample period are not drawn the unique normal distribution. Thus it would be challenging to apply numerical-analytic method.

From the observed data we can assume that $S_{\max} = 28{,}10, S_{\min} = 24{,}15$. The analysis of the data showed that the average daily change of price is $\Delta = 0{,}21$. So the state space is consisted of 22 elements. Statistical evaluations of non-central moments of the stock prices moving up and down are the following:

$$m_1^U = 2{,}2402; \; m_2^U = 11{,}6950; \; m_3^U = 85{,}7250;$$
$$m_1^D = 1{,}8321; m_2^D = 6{,}9676; \; m_3^D = 37{,}4190.$$

Corresponding parameters of approximating density function are calculated according to formulas given in section 3. The parameters are the following:

$$\mu_1^U = 0{,}4014; \; \mu_2^U = 1{,}2384; \; \mathrm{p}_1^{\mathrm{U}} = 0{,}2373;$$
$$\mu_1^D = 0{,}5381; \mu_2^D = 0{,}3358; \; \mathrm{p}_1^{\mathrm{D}} = 0{,}0088;$$

The dynamics of stock price movements was described in the event language presented in section 4. The developed software using the description as input data automatically generates all the possible states (the set of stock prices) and transition rates among them, calculates steady state probabilities of asset prices. The calculated probability distribution is given in the Table 1.

Table 1. Probability distribution of asset prices

State	24,15	24,36	24,57	24,78	24,99	25,2	25,41
Probability	0,0015	0,0017	0.0018	0.0018	0.028	0.0042	0.0079

State	25,41	25,62	25,83	26,04	26,25	26,46	26,67
Probability	0.0079	0.0154	0.0324	0.1609	0.1516	0.1187	0.1002

State	26,88	27,09	27,30	27,51	27,72	27,93	28,10
Probability	0.0860	0.0732	0.0613	0.0523	0.0453	0.0387	0.0423

The distribution can be used for evaluation of statistical measures of the stock prices and pricing derivative securities.

6 Conclusion

This paper gives a continuous time Markov chain model for asset dynamics. It allows solving of a large size problem. The model can be applied for pricing of option financial products working in continuous time and with countable number of possible values for the imbedded asset, which is always the case from the numerical point of view. The main interest of this model is that it works even when there are possibilities of arbitrage, i.e. the most frequent cases. Further research will be devoted for determining the risk neutral measure and pricing derivative securities based on the continuous time Markov chain model.

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