Darwin Approximation to Maxwell's Equations

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Abstract. Maxwell's equations are the foundation of electromagnetics, its extensive applications make computational electromagnetics a new rapidly growing subject. However, since the electric and the magnetic fields are coupling in the Maxwell system, the numerical solution of the full system of Maxwell's equations is extremely expensive in terms of computer time. Darwin model, which decouple the coupled system, is such a good approximation to Maxwell's equations. In three dimensional space case, it neglects the transverse component of the displacement current; while in two dimensional space case, Darwin model is equivalent to TE, TM models. For these reasons, it is worthwhile to investigate Darwin model, and a lot of works have been done to it. In this paper, we review the advance in the Darwin model and introduce some of our new results about the exterior problems of Darwin model and its numerical solutions with infinite elements methods.

Keywords: Maxwell's equations, Darwin model, infinite element method, exterior problem.

1 Introduction

More and more modeling and simulation of electronic devices are related to the computation of electromagnetics which becomes a very important branch in the present computational science. However, since the multiphysics fields of electric and magnetic fields are coupling together in the Maxwell's equations, there are many challenges in dealing with the coupling system directly, and the numerical solutions may be very expensive in terms of the computational cost. On the other hand, for some problems, such as the simulations of charged particle beams when no high frequency phenomenon or no rapid current charge occurs, Darwin model, which appears as a correction of the quasi-electrostatic model, including the electric fields generated by magnetic induction, is a good approximation to Maxwell's equations. This model is obtained by neglecting the solenoidal part of the displacement current in the Maxwell's equations, and exhibits an elliptic character with infinite propagation speed. Thus, the multiphysics fields

of electricity and magnetism are separated in several single physical field systems which are easy to solve.

It has been proved in 1992 by P.Degond and P.A.Raviart [1] that Darwin model in three dimensional cases approximates Maxwell's equations up to the second order of the dimensionless parameter $\eta = \bar{v}/c$ for magnetic field $\mathbf{B}$ and to the third order of η for electric field $\mathbf{E}$, provided that η is small, where $\bar{v}$ is characteristic velocity and c is light velocity. While in two dimension cases, Darwin model is equivalent to TE, TM models without any approximation [15].

Some earlier investigations have been done by D.W.Hewett and C.W.Nielson etc, please refer to [2,3,4] for a derivation of the Darwin model from the plasma point and its numerical implementation by streamline methods in two dimensional bounded domains.

In 1995, P.Ciarlet and J.Zou [5] have studied the $\mathbf{H}(\text{curl}; \Omega)$ and $\mathbf{H}(\text{curl}, \text{div}; \Omega)$ variational formulations for the Darwin model in three dimensional bounded domains, and proved the well-posedness of the variational systems.

Since quite a number of problems in applications are related to unbounded domains, L.Ying and F.Li [6] in 2002 considered Darwin model of electric field with boundary value conditions in two dimensional exterior domains.

Recently, the Darwin model of magnetic field has been considered by C.Liao and L.Ying [7], the results are similar. An analysis of the Darwin model of approximation to Maxwell's equations in 3-D unbounded domains was given in [16].

At the same time, three dimensional exterior problem of Darwin model of both electric and magnetic fields and the numerical computation has been studied by N.Fang and L.Ying [8].

The following contents are organized as: In Section 2 we recall the relationship of Maxwell's equations and Darwin model. While in Section 3 we review the work in the foretime and introduce our recent results. And a short conclusion is given in Section 4.

2 Maxwell's Equations and Darwin Model

Maxwell's equations in vacuum are of the following form in the space-time domain $\Omega \times (0, T)$, where Ω is an open simply connected domain Ω^c in $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$. (All the variables and function spaces in bold are denoted by vector ones):

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \quad (\text{Ampère law}) \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \quad (\text{Faraday law}) \quad (2)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad (\text{Gauss' law}) \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{no source assumption of magnetic}) \quad (4)$$

Where $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ denote the electric field and the magnetic field respectively, and $\rho = \rho(\mathbf{x}, t)$, $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ are respectively the charge and current densities satisfying

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (\text{charges' conservation law}). \quad (5)$$

The positive constants c , ε_0 , and μ_0 are respectively the light velocity, the electric permittivity, and the magnetic permeability in vacuum, which are related by

$$\varepsilon_0 \mu_0 c^2 = 1.$$

The relationship of Darwin model and Maxwell's equations are discussed in [1] when adding some initial conditions and assuming $\partial\Omega$ a perfect conductor. Since the electric field $\mathbf{E}$ can be decomposed into a sum of some transverse component $\mathbf{E}_T$ and some longitudinal component $\mathbf{E}_L$. Substituting the decomposition into equation (1) and neglecting the term $\frac{\partial \mathbf{E}_T}{\partial t}$, one can get

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}_L}{\partial t}. \quad (6)$$

Then the Maxwell's equations (1) – (4) will be reduced to the following Darwin model, see [1,14] for deduction details:

(i) $\mathbf{E}_L = -\nabla\Phi$, and Φ is the solution of the following Dirichlet problem:

$$-\Delta\Phi = \frac{\rho}{\varepsilon_0}, \text{ in } \Omega, \quad (7)$$

$$\Phi = C(t), \text{ on } \partial\Omega, \quad (8)$$

$$\Phi \rightarrow 0, \text{ at infinity}. \quad (9)$$

Where $C(t)$ is an arbitrary function only depend on time t if Ω are bounded domains. If Ω represents exterior domains, $C(t)$ is a constant independent of time t , for example $C(t) = 1$ if satisfying the boundary assumptions in [5].

(ii) $\mathbf{B}$ satisfies

$$-\Delta\mathbf{B} = \mu_0 \nabla \times \mathbf{J}, \text{ in } \Omega, \quad (10)$$

$$\nabla \cdot \mathbf{B} = 0, \text{ in } \Omega, \quad (11)$$

$$\mathbf{B} \cdot \mathbf{n} = \mathbf{B}_0 \cdot \mathbf{n}, \text{ on } \partial\Omega, \quad (12)$$

$$(\nabla \times \mathbf{B}) \times \mathbf{n} = \mu_0 \mathbf{J} \times \mathbf{n}, \text{ on } \partial\Omega. \quad (13)$$

(iii) $\mathbf{E}_T$ satisfies

$$\Delta\mathbf{E}_T = \frac{\partial}{\partial t} \nabla \times \mathbf{B}, \text{ in } \Omega, \quad (14)$$

$$\nabla \cdot \mathbf{E}_T = 0, \text{ in } \Omega, \quad (15)$$

$$\mathbf{E}_T \times \mathbf{n} = 0, \text{ on } \partial\Omega, \quad (16)$$

$$\langle \mathbf{E}_T \cdot \mathbf{n}, 1 \rangle = 0, \text{ on } \partial\Omega. \quad (17)$$

The coupling multiphysics fields of Maxwell's equations are reduced to three single physical field systems. $\mathbf{E}_L$ relies on problem (i), a simple Poisson equation. $\mathbf{B}$ are solved by problem (ii), then plugging it into problem (iii), one can get $\mathbf{E}_T$. All the mathematical challenges to Darwin model are how to solve the problems (ii) and (iii).

3 Darwin Model and Its Numerical Methods

Both problem (ii) and problem (iii) contain equation $\nabla \cdot \mathbf{u} = 0$, i.e. (11) and (15), and it is very difficult to deal in numerical computations. All the numerical studies of Darwin model to problems (ii) and (iii) in fact were carried on to this point and gave their own ways to manage it.

D.W.Hewett and J.K.Boyd analyzed the solutions of Darwin model by stream-line method in [3], and they discussed methods for calculating the magnetic field without formal vector decomposition and offered a new procedure for finding the inductive electric field. As a consequence, the numerical efforts required for each of the field time-steps are reduced, and more importantly, the needs to specify several nonintuitive boundary conditions are eliminated.

In P.Degond and P.A.Raviart's paper [1], they studied appropriate decompositions of vector fields which give rise to the well-posedness of the Darwin model. Besides, they obtained a by-product, that is the uniqueness of the variational formulations of the elliptic boundary value problem of Darwin model.

Later works are processed to tackle Darwin model through mixed variational methods by using Babuska and Brezzi's saddle point theory [12], which is said:

Let $\mathbf{X}$ and Q be two Hilbert spaces with norms $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_Q$ respectively, $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ be two continuous bilinear forms defined respectively on $\mathbf{X} \times \mathbf{X}$ and $\mathbf{X} \times Q$, and $f(\cdot)$ and $g(\cdot)$ be two continuous linear forms defined respectively on $\mathbf{X}$ and Q . Then the problem : Find $(\mathbf{u}, p) \in (\mathbf{X}, Q)$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= f(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}, \\ b(\mathbf{u}, q) &= g(q), \quad \forall q \in Q \end{aligned} \quad (18)$$

is called a saddle point problem. Let $\mathbf{V}$ be a closed subspace of $\mathbf{X}$ defined as

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X}; b(\mathbf{v}, q) = 0, \quad \forall q \in Q\}.$$

Assume that there exist two positive constants α and β such that

$$a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|_{\mathbf{X}}^2, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (\mathbf{V}\text{-ellipticity}) \quad (19)$$

and

$$\sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{X}}} \geq \beta \|q\|_Q, \quad \forall q \in Q. \quad (\text{inf-sup condition}) \quad (20)$$

Then there exists a unique solution to the saddle point problem (18).

So the key point turns to investigate the mixed variational formulations of the Darwin model and to verify the $\mathbf{V}$ -ellipticity and the inf-sup condition.

The details to discuss the variational formulations of Darwin model were given by P.Ciarlet, JR and J.Zou in [5]. They have studied the $\mathbf{H}(\mathbf{curl}; \Omega)$ and $\mathbf{H}(\mathbf{curl}, \text{div}; \Omega)$ variational formulations for Darwin model in three dimensions, and proved the well-posedness. Nedelec's finite element method is used to solve the $\mathbf{H}(\mathbf{curl}; \Omega)$ variational formulation where $\nabla \cdot \mathbf{u} = 0$ is naturally satisfied in such finite element method. Moreover, they discussed a standard finite element

method of so called Hood-Taylor finite element [12] for the $\mathbf{H}(\mathbf{curl}, \text{div}; \Omega)$ variational formulation. Convergence and the error estimates were obtained without requiring the physical domains to be convex. They did not give numerical experiments.

Since quite a number of problems in applications are related to exterior domains, L.Ying and F.Li in 2002 considered the exterior problem of Darwin model in [6]. They first considered Darwin model of electric field with boundary value conditions in two dimensional exterior domains. In order to show the coercivity in weighted space $\mathbf{W}$ whose elements belong to $\mathbf{H}(\mathbf{curl}, \text{div}; \Omega)$ in finite domain while in the infinity distance belong to weighted space $\mathbf{H}^{1,*}(\Omega)$ (see [9] for a definition), they spent a lot of energy and at last made clear that

$$\|\mathbf{u}\|_* = \{\|\nabla \times \mathbf{u}\|_0^2 + \|\nabla \cdot \mathbf{u}\|_0^2 + \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\partial\Omega}\}^{\frac{1}{2}}$$

is only a semi-norm on $\mathbf{W}$. With analytic function theory, they got a null space $\mathbf{V}_0$ of two degrees of freedom. Then they were able to establish the variational formulation and proved its well-posedness in the quotient space $\mathbf{W}/\mathbf{V}_0$. Numerical examples by infinite element methods were given as well, and $P_2 - P_0$ elements and square exterior domain with triangle meshes were chosen to implement the calculations. They got a so-called transfer matrix [10] in virtue of Stoke's equations since the solution of Darwin model was just that of Stoke's equation with appropriate boundary value and inhomogeneous terms. Numerical results illustrated three-order convergence under L^2 -norm and one-order convergence under $\|\cdot\|_*$ norm, which was quite good in accordance with the theoretical error estimate results.

Later on, the magnetic field of Darwin model has been considered by C.Liao and L.Ying [7]. The results are similar, and the difference is that there are three degrees of freedom in the null space because the boundary conditions are different. Although differing from a function in $\mathbf{V}_0$, they manage to prove the uniqueness of the solutions if imposing a boundary condition at the infinity $\mathbf{u}|_{|x| \rightarrow \infty} = 0$ for Darwin model. This was once left as an open problem in [6].

Meanwhile, the three dimensional exterior problem of Darwin model and its numerical computation has been studied by N.Fang and L.Ying [8]. Quite different from the two dimensional exterior domains cases, $\|\cdot\|_*$ is just a norm in three dimensional case, such fact was proved through potential theory. So the well-posedness of the variational formulations were able to proved directly in $\mathbf{W}$. Moreover, for the need of investigating the Darwin model, they proved an important decomposition that any function in exterior domain Ω belonging to $\mathbf{L}^2(\Omega)$ can be decomposed to the sum of a gradient function and a rotational function, i.e. *For any $\mathbf{f} \in \mathbf{L}^2(\Omega)$, it has the decomposition $\mathbf{f} = \nabla\phi + \nabla \times \mathbf{u}$, where $\phi \in H_c = \{\phi \in H^{1,*}(\Omega); \nabla\phi \times \mathbf{n} = 0\}$, $\mathbf{u} \in \mathbf{B}_0$* (see [8] for the definitions). Numerical experiments with infinite element method are carried on for axisymmetric fields. The difficulty is that the variational formulation has to be considered in some weighted Sobolev spaces, the results show the convergence once again.

Fig. 1. Sketch maps of first layer grids in two spacial dimensions and three dimensional axisymmetric space cases

The infinite element method, which is suitable particularly for unbounded domain problems and singular problems, has its complete academic foundations and wide applications [10]. One great character of the infinite element method distinguished it with other methods is to design a similar triangulation with infinite grids for the question domain. The similar triangulation, which divides the interested domain into layers one by one with a proportionality, is the same as that of classical finite element methods in each layer. The outstanding point is that the infinite element method can approximate an exterior problem to the infinite or a singular problem to the singular position. Of course, it will get a discrete system with infinite algebraic equations. However, because in each layer the stiffness matrices are similar in a proportion, there exists a real transfer matrix which can transfer the values from one layer to the other layers. Thus, the whole computational workload for the infinite element method is reduced to solve a classical finite element in the first layer and to find the transfer matrix. For many exterior or singular problems, such as elliptic problems and singular problems, it is economical and efficient to use the infinite element method to find the solutions.

The following are parts of the numerical results of Darwin model in two spacial dimensions [6,7] and three dimensional axisymmetric space cases with infinite element methods [8]. Fig.1 are sketch maps of the first layer grids of the infinite element triangulations in two spacial dimension and three dimensional axisymmetric domains.

Example 1. (Tested by F.Li [6]) Two dimensional Darwin model of electric field with exact solutions

$$\mathbf{u} = \left(\frac{\cos \theta}{r}, \frac{-\sin \theta}{r} \right).$$

It calculates the errors on the $(m+1)$ layers, where N is the number of nodes on Γ_0 . One can find in Table 1 more than two order and nearly one order convergence under corresponding norms which obeys the theoretical results.

Table 1. $\xi = 1.20$, $m=20$, errors of the electricity field Darwin model

N	$\ \mathbf{u}_1 - \mathbf{u}_{1h}\ _0$	order	$\ \mathbf{u}_2 - \mathbf{u}_{2h}\ _0$	order	$\ \mathbf{u} - \mathbf{u}_h\ _*$	order
16	7.097883E-02		7.104025E-02		0.705941	
32	9.723456E-03	2.8678	9.723475E-03	2.8691	0.385342	0.8734
64	1.591538E-03	2.6110	1.142021E-03	3.0899	0.196713	0.9700
128	1.341634E-04	3.5684	1.491097E-04	2.9371	9.862373E-02	0.9961

Example 2. (Tested by C.Liao [7]) Two dimensional Darwin model of magnetic field with exact solutions

$$u_1 = \frac{x^3 - 3xy^2}{(x^3 - 3xy^2)^2 + (3x^2y - y^3)^2}, \quad u_2 = \frac{3x^2y - y^3}{(x^3 - 3xy^2)^2 + (3x^2y - y^3)^2},$$

the numerical results are similar as Example 1, see Table 2.

Table 2. $\xi = 1.10$, $m=20$, errors of the magnetic field Darwin model

N	$\ \mathbf{u}_1 - \mathbf{u}_{1h}\ _0$	order	$\ \mathbf{u}_2 - \mathbf{u}_{2h}\ _0$	order	$\ \mathbf{u} - \mathbf{u}_h\ _*$	order
16	0.203937		0.205719		1.403172	
32	2.987612E-02	2.7712	3.101684E-02	2.7696	0.543875	1.3674
64	4.002932E-03	2.8999	4.170768E-03	2.8547	0.180869	1.5884

Example 3. (Tested by N.Fang [8]) Three dimensional axisymmetric cases, an exact solution expressed in cylindrical coordinate is

$$\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) = \left(-\frac{3}{16} \frac{rz}{(r^2 + z^2)^{5/2}}, \frac{1}{16} \frac{r^2 - 2z^2}{(r^2 + z^2)^{5/2}}\right). \quad (21)$$

Table 3. $\xi = 1.20$, $m=20$, errors of the electric field Darwin model

N	ξ	$\ \mathbf{E} - \mathbf{E}_h\ _{L^2_\uparrow}$	order	$\ \mathbf{E} - \mathbf{E}_h\ _\uparrow$	order
17	1.200	0.121738		1.087437	
33	1.100	0.024777	2.29669	0.457862	1.24794
65	1.050	6.38095E-03	1.95716	0.226430	1.01585
129	1.025	1.84555E-03	1.78972	0.125478	0.85163

Table 4. $\xi = 1.20$, $m=20$, errors of the magnetic field Darwin model

N	ξ	$\ \tilde{\mathbf{B}} - \tilde{\mathbf{B}}_h\ _{L^2_\uparrow}$	order	$\ \tilde{\mathbf{B}} - \tilde{\mathbf{B}}_h\ _\uparrow$	order
17	1.200	0.512389		2.384971	
33	1.100	0.118179	2.11627	1.081343	1.14115
65	1.050	3.03803E-02	1.93981	0.520884	1.05379
129	1.025	8.47473E-03	1.84189	0.267462	0.96163

Table 5. Errors with different ξ' s with $N = 33$ in domain S_2

ξ	$\ \mathbf{E} - \mathbf{E}_h\ _{L^2_\dagger}$	$\ \mathbf{E} - \mathbf{E}_h\ _\dagger$	$\ \mathbf{B} - \mathbf{B}_h\ _{L^2_\dagger}$	$\ \mathbf{B} - \mathbf{B}_h\ _\dagger$
2.60	0.989638	3.374672	4.513487	7.374385
1.85	0.330322	2.264038	2.163367	5.536487
1.54	0.120657	1.743531	0.058450	4.115571
1.20	0.024218	1.087437	0.014065	2.384973
1.05	0.027387	0.845684	0.011470	1.314983
1.01	0.020547	0.837463	0.009477	1.237863

Table 6. The solved components of electric field on Γ_0 with $N = 33$ and different ξ' s

nodes	$\xi = 1.80$	$\xi = 1.50$	$\xi = 1.20$	$\xi = 1.10$	$\xi = 1.01$	exact values
1	-0.1210545	-0.1257055	-0.1260910*	-0.1260891	-0.1215592	-0.1250000
2	-0.1126088	-0.1176541	-0.1190080	-0.1186574	-0.1145254	-0.1193081
3	-0.0975158	-0.1024953	-0.1040261	-0.1037537	-0.9970447	-0.1040640
4	-0.0797613	-0.0829016	-0.0837716	-0.0827922	-0.0791954	-0.0836353
5	-0.0595527	-0.0615719	-0.0626015	-0.0618062	-0.0603292	-0.0626099
6	-0.0429919	-0.0439044	-0.0443304	-0.0437183	-0.4435603	-0.0441075
7	-0.0281114	-0.0286289	-0.0294198	-0.0290186	-0.0297962	-0.0294402
8	-0.0186065	-0.0186243	-0.0187584	-0.0184425	-0.0189317	-0.0186243
10	0.0389879	0.0396177	0.0397538	0.0396909	0.0399477	0.0396062
11	0.0444103	0.0457516	0.0461985	0.0462046	0.0464945	0.0460800
12	0.0480863	0.0503425	0.0512331	0.0513768	0.0512298	0.0513873
13	0.0497539	0.0527489	0.0539654	0.0542693	0.0535041	0.0536656
14	0.0451329	0.0484963	0.0500021	0.0505106	0.0472572	0.0506029
15	0.0363879	0.0395276	0.0407025	0.0410631	0.0272850	0.0402827
16	0.0194135	0.0214026	0.0221285	0.0225052	0.0036119	0.0225464
17	-0.0001567	0.0002899	0.0001114	0.0003259	0.0014535	0.0000000
18	-0.0204573	-0.2158120	-0.0222988	-0.0225053	-0.0006429*	-0.0225464
19	-0.0372877	-0.0394444	-0.0404561	-0.0403204	-0.0294834	-0.0402827
20	-0.0466633	-0.0492025	-0.0503285	-0.0502459	-0.049193	-0.0506029
21	-0.0505969	-0.0528761	-0.0537361	-0.0537916	-0.0563653	-0.0536656
22	-0.0487024	-0.0505964	-0.0513363	-0.0514396	-0.0524078	-0.0513873
23	-0.0442433	-0.0454351	-0.0460040	-0.0463146	-0.0465934	-0.0460800
24	-0.0387989	-0.0394013	-0.0396301	-0.0397049	-0.039857	-0.0396062
26	-0.0190566	-0.0192702	-0.0190187	-0.0188858	-0.0157653	-0.0186243
27	-0.0283864	-0.0291424	-0.0291973	-0.0292054	-0.0254894	-0.0294402
28	-0.0424493	-0.0442339	-0.0444911	-0.4438434	-0.0412477	-0.0441075
29	-0.0576513	-0.0609854	-0.0622284	-0.0623327	-0.0616994	-0.0626099
30	-0.0758385	-0.0813233	-0.0834862	-0.0835264	-0.0857989	-0.0836353
31	-0.0920192	-0.0999494	-0.1037470	-0.1041031	-0.1100688	-0.1040640
32	-0.1040798	-0.1137892*	-0.1184702	-0.1188476	-0.1290048	-0.1193081
33	-0.1094091*	-0.1204418	-0.1260022	-0.1264943*	-0.1389577	-0.1250000
Max error*	0.12473	4.4154E-02	1.1926E-02	1.1950E-02	0.17522	

Table 3 and Table 4 show that about first order convergence under the norms $\|\cdot\|_{\dagger}$ for both the magnetic and the electric field are obtained. Table 5 lists the errors in the domain $S_2 = \{\mathbf{x}; 1 \leq |r|, |z| \leq 32\}$, and the error tends a level when ξ decrease. Here ξ is the proportionality constant of infinite element triangulation, and m is the numbers of layers been calculated.

Example 4. (Tested by N.Fang [8]) Comparisons of local quantities of interest are made in Table 6 for the axisymmetric electric field case. Axisymmetric magnetic field and two dimensional space cases are similar [6,7,8]. The numerical experiments illustrate that one can find “good” solutions by refining the meshes and decrease the proportion ξ at the same time.

4 Conclusions

It gives at least a way to deal with some multiphysics models by decomposing the coupling system. Darwin model is such a successful example which approximates Maxwell's equations well enough. Hereto, Darwin models are thoroughly studied both in bounded domains and exterior domains especially by variational methods. Our investigations in establishing the general theory are mostly about the exterior problems. Suitable tools like analytic functions theory, potential theory and weighted Sobolev spaces are utilized to finish the theoretical analysis. Besides, infinite element methods are effective to simulate the models in unbounded domains, this was verified by the numerical experiments. Of course, as an approximate model to Maxwell's equations, Darwin model are worthy of further study. It leaves to combine the problems of (i), (ii) and (iii) together. Some related works such as adaptive finite elements to Darwin model are being studied by the authors.

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