Integers Powers of Certain Asymmetric Matrices

Roman Wituła and Damian Słota

Institute of Mathematics
Silesian University of Technology
Kaszubska 23, 44-100 Gliwice, Poland
{roman.witula,damian.slota}@polsl.pl

Abstract. In this paper we derive the recurrent formulae for any integers powers of certain asymmetric matrices five order which covers also constans tridiagonal matrices and some asymmetric (and symmetric) pentadiagonal matrices. Since we find also the Binet's formulae for the elements of the powers of these matrices, it is possible to operate with powers series of these matrices. We note that this method is alternative to the Jordan method decomposition.

Keywords: tridiagonal matrices, asymmetric matrices.

1 Introduction

The scope of the paper is to determine the recurrent formulae for the powers of certain asymmetric matrices belonging to the class $M_{5 \times 5}(\mathbb{C})$, involving constans tridiagonal matrices that have a very wide application. The recurrence method, which is an alternative to the Jordan's decomposition method, Leonard's algorithm, Putzer's algorithm and the method of characteristic polynomial [1,2,3,4,5,6,7], seems fast and facilitating the definition of integers powers of any matrices. The paper is a follow up on [8] (which is inspired by [9,10], see also [11]), where the cases of the third and fourth order asymmetric matrices are discussed and where also the complex powers of these matrices are defined. Apart from the proposed formulae, the authors also present a computational example of the application of this procedure.

2 Basic Recurrence Formulae

Let us set now:

$$\mathbf{G}_5(a, b, c, d, e, f, g, A, B, y) := \begin{bmatrix} a & yb & y^2c & y^3e & y^4g \\ b & A & yd & y^2f & y^3e \\ c & d & B & yd & y^2c \\ e & f & d & A & yb \\ g & e & c & b & a \end{bmatrix} \quad (1)$$

for any $a, b, c, d, e, f, g, A, B, y, \in \mathbb{C}$.

Lemma 1. Let $a_1, b_1, c_1, d_1, e_1, f_1, g_1, A_1, B_1, y, \in \mathbb{C}$. If

$$\begin{aligned}d_1 &= b_1 + y e_1, \\f_1 &= c_1 + y g_1, \\A_1 &= a_1 + y c_1, \\B_1 &= a_1 + y c_1 + y^2 g_1,\end{aligned}$$

then we get

$$\begin{aligned}\left(\mathbf{G}_5(a_1, b_1, c_1, d_1, e_1, f_1, g_1, A_1, B_1, y)\right)^k &= \\&= \mathbf{G}_5(a_k, b_k, c_k, d_k, e_k, f_k, g_k, A_k, B_k, y)\end{aligned}\quad (2)$$

for every $k \in \mathbb{N}$, where the following recurrence formulae hold:

$$a_{k+1} = a_1 a_k + y b_1 b_k + y^2 c_1 c_k + y^3 e_1 e_k + y^4 g_1 g_k, \quad (3)$$

$$b_{k+1} = b_1 a_k + A_1 b_k + y d_1 c_k + y^2 f_1 e_k + y^3 e_1 g_k, \quad (4)$$

$$\begin{aligned}c_{k+1} &= c_1 a_k + B_1 c_k + y^2 c_1 g_k + d_1 d_k \\&= B_1 c_k + B_k c_1 - y c_1 c_k + d_1 d_k,\end{aligned}\quad (5)$$

$$e_{k+1} = e_1 a_k + f_1 b_k + d_1 c_k + A_1 e_k + y b_1 g_k, \quad (6)$$

$$g_{k+1} = g_1 a_k + e_1 b_k + c_1 c_k + b_1 e_k + a_1 g_k, \quad (7)$$

$$d_k = b_k + y e_k, \quad (8)$$

$$f_k = c_k + y g_k, \quad (9)$$

$$A_k = a_k + y c_k, \quad (10)$$

$$B_k = a_k + y c_k + y^2 g_k. \quad (11)$$

We note that if additionally we have $c_1 = g_1 = e_1 = 0$ then $G_5(a_1, \dots, y)$ is a tridiagonal matrix.

The proof of the lemma by an easy induction arguments follows.

Now we want to find the Binet's formulae for the elements a_k, b_k, c_k, e_k and g_k . To this aim it is sufficient to designate twenty five coefficients ξ_i for $\xi \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$ and $i = 1, 2, 3, 4, 5$, for which the following system of equations hold:

$$a_k = \alpha_1 t^k + \beta_1 u^k + \gamma_1 v^k + \delta_1 w^k + \varepsilon_1 z^k, \quad (12)$$

$$y^{1/2} b_k = \alpha_2 t^k + \beta_2 u^k + \gamma_2 v^k + \delta_2 w^k + \varepsilon_2 z^k, \quad (13)$$

$$y c_k = \alpha_3 t^k + \beta_3 u^k + \gamma_3 v^k + \delta_3 w^k + \varepsilon_3 z^k, \quad (14)$$

$$y^{3/2} e_k = \alpha_4 t^k + \beta_4 u^k + \gamma_4 v^k + \delta_4 w^k + \varepsilon_4 z^k, \quad (15)$$

$$y^2 g_k = \alpha_5 t^k + \beta_5 u^k + \gamma_5 v^k + \delta_5 w^k + \varepsilon_5 z^k, \quad (16)$$

for every $k \in \mathbb{N}$ and for some $t, u, v, w, z \in \mathbb{C}$ independent on k .

From identities (3)–(7) and (12)–(16) if we compare the coefficients of powers u^{k+1} , v^{k+1} , w^{k+1} and z^{k+1} it is easily established the following system of equations

$$\xi_1 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2, \quad (17)$$

$$\xi_2 = 2\xi_1\xi_2 + 2\xi_2\xi_3 + 2\xi_3\xi_4 + 2\xi_4\xi_5, \quad (18)$$

$$\xi_3 = 2\xi_1\xi_3 + 2\xi_2\xi_4 + 2\xi_3\xi_5 + \xi_2^2 + \xi_3^2 + \xi_4^2, \quad (19)$$

$$\xi_4 = 2\xi_1\xi_4 + 2\xi_2\xi_3 + 2\xi_3\xi_4 + 2\xi_2\xi_5, \quad (20)$$

$$\xi_5 = 2\xi_1\xi_5 + 2\xi_2\xi_4 + \xi_3^2, \quad (21)$$

for every $\xi \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$.

Sketch of solution of the system (17)–(21):

Subtracting equations: (21) from (17) and next (18) from (20) we obtain respectively:

$$\xi_1 - \xi_5 = (\xi_1 - \xi_5)^2 + (\xi_2 - \xi_4)^2, \quad (22)$$

$$\xi_2 - \xi_4 = 2\xi_1(\xi_2 - \xi_4) + 2\xi_5(\xi_4 - \xi_2) = 2(\xi_2 - \xi_4)(\xi_1 - \xi_5),$$

i.e.,

$$(\xi_2 - \xi_4)(1 - 2(\xi_1 - \xi_5)) = 0. \quad (23)$$

From (23) we have, that either $\xi_2 - \xi_4 = 0$ or $\xi_1 - \xi_5 = \frac{1}{2}$. If $\xi_2 = \xi_4$ then by (22) either $\xi_1 = \xi_5$ or $\xi_1 - \xi_5 = 1$. On the other hand, if $\xi_1 - \xi_5 = \frac{1}{2}$, then by (22) $\xi_2 - \xi_4 = \pm\frac{1}{2}$. Finally, the equations system (17)–(21) is extended to include one of the following four equations:

1. $\xi_2 = \xi_4$ and $\xi_1 = \xi_5$,
2. $\xi_2 = \xi_4$ and $\xi_1 - \xi_5 = 1$,
3. $\xi_1 - \xi_5 = \frac{1}{2}$ and $\xi_2 - \xi_4 = \frac{1}{2}$,
4. $\xi_1 - \xi_5 = \frac{1}{2}$ and $\xi_2 - \xi_4 = -\frac{1}{2}$,

reducing the number of solutions to 28 vectors from $\mathbb{R}^5$ (all calculations were performed in *Mathematica*). Surely, out of the 28 vectors, we are interested in only five different ones. It should be indicated that in view of equations (3)–(7), the required vectors must comply with the following orthogonal conditions:

$$\sum_{i=1}^5 \zeta_i \eta_i = \sum_{i=1}^5 \zeta_i \eta_{5-i} = 0$$

for every $\zeta, \eta \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\zeta \neq \eta$. From this point, it is easy to derive the following solution:

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \varepsilon_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \varepsilon_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \delta_5 & \varepsilon_5 \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & \frac{1}{12} \\ \frac{-1}{4\sqrt{3}} & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4\sqrt{3}} \\ \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} \\ \frac{-1}{4\sqrt{3}} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4\sqrt{3}} \\ \frac{1}{12} & -\frac{1}{4} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{12} \end{bmatrix}. \quad (24)$$

Moreover, we can generate the following formulae

$$t = a_1 - b_1 \sqrt{3y} + 2c_1 y + g_1 y^2 - e_1 \sqrt{3y^3}, \quad (25)$$

$$u = a_1 + b_1 \sqrt{y} - g_1 y^2 - e_1 \sqrt{y^3}, \quad (26)$$

$$v = a_1 - c_1 y + g_1 y^2, \quad (27)$$

$$w = a_1 - b_1 \sqrt{y} - g_1 y^2 + e_1 \sqrt{y^3}, \quad (28)$$

$$z = a_1 + b_1 \sqrt{3y} + 2c_1 y + g_1 y^2 + e_1 \sqrt{3y^3}. \quad (29)$$

Corollary 1. *We have*

$$\begin{bmatrix} a & c & 0 & 0 & 0 \\ b & a & c & 0 & 0 \\ 0 & b & a & c & 0 \\ 0 & 0 & b & a & c \\ 0 & 0 & 0 & b & a \end{bmatrix}^r = \mathbf{G}_5(\alpha_r, \beta_r, \gamma_r, \delta_r, \varepsilon_r, f_r, g_r, A_r, B_r, \frac{c}{b}), \quad (30)$$

where

$$\alpha_r = \frac{1}{12} t^r + \frac{1}{4} u^r + \frac{1}{3} v^r + \frac{1}{4} w^r + \frac{1}{12} z^r, \quad (31)$$

$$y^{1/2} \beta_r = \frac{-1}{4\sqrt{3}} t^r + \frac{1}{4} u^r - \frac{1}{4} w^r + \frac{1}{4\sqrt{3}} z^r, \quad (32)$$

$$y \gamma_r = \frac{1}{6} t^r - \frac{1}{3} v^r + \frac{1}{6} z^r, \quad (33)$$

$$y^{3/2} \varepsilon_r = \frac{-1}{4\sqrt{3}} t^r - \frac{1}{4} u^r + \frac{1}{4} w^r + \frac{1}{4\sqrt{3}} z^r, \quad (34)$$

$$y^2 g_r = \frac{1}{12} t^r - \frac{1}{4} u^r + \frac{1}{3} v^r - \frac{1}{4} w^r + \frac{1}{12} z^r, \quad (35)$$

$$y^{1/2} \delta_r = y^{1/2} \beta_r + y^{3/2} \varepsilon_r = \frac{\sqrt{3}}{6} (z^r - t^r), \quad (36)$$

$$4y f_r = 4(y \gamma_r + y^2 g_r) = t^r - u^r - w^r + z^r, \quad (37)$$

$$4A_r = 4(\alpha_r + y \gamma_r) = t^r + u^r + w^r + z^r, \quad (38)$$

$$3B_r = 3(\alpha_r + y \gamma_r + y^2 g_r) = t^r + v^r + z^r, \quad (39)$$

and

$$t = a - \sqrt{3bc},$$

$$u = a + \sqrt{bc},$$

$$v = a,$$

$$w = a - \sqrt{bc},$$

$$z = a + \sqrt{3bc}.$$

Remark 1. All formulas (2)–(16) and (30)–(39) hold true also for all $k, r \in \mathbb{Z}$, since $\mathbf{G}_5(a_1, b_1, \dots, g_1, A_1, B_1, y)$ is the diagonal matrix (see also [8] for more

comments). The powers in the respective formulas should be chosen in the following way:

$$x^r := \exp(r(\ln|x| + i \operatorname{Arg} x)).$$

3 Example

Now the numerical example of application of our formulae will be given. Let us set: $a_1 = 1$, $b_1 = 1$, $c_1 = -1$, $e_1 = -1$, $g_1 = 1$, $d_1 = 1 - i = \sqrt{2}e^{-i\pi/4}$, $f_1 = -1 + i = \sqrt{2}e^{i3\pi/4}$, $A_1 = 1 - i = \sqrt{2}e^{-i\pi/4}$, $B_1 = -i$, $y = i$. For these values, we have

$$\mathbf{G}_5(a_1, b_1, \dots, B_1, y) = \begin{bmatrix} 1 & i & 1 & i & 1 \\ 1 & 1-i & 1+i & 1-i & i \\ -1 & 1-i & -i & 1+i & 1 \\ -1 & -1+i & 1-i & 1-i & i \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix}.$$

By (25)–(29) we get

$$\begin{aligned} t &= 1 - \sqrt{3}i - 2i - 1 + \sqrt{-3}i \\ &= -\exp\left(\frac{1}{2}(\ln 3 + i\frac{\pi}{2})\right) - 2i + \exp\left(\frac{1}{2}(\ln 3 + i\frac{3\pi}{2})\right) \\ &= -\sqrt{3}\exp\left(i\frac{\pi}{4}\right) - 2i + \sqrt{3}\exp\left(i\frac{3\pi}{4}\right) \\ &= -\sqrt{6} - 2i, \\ u &= 2 + i\sqrt{2}, \\ v &= i, \\ w &= 2 - i\sqrt{2}, \\ z &= \sqrt{6} - 2i. \end{aligned}$$

Then by (2) we have

$$\mathbf{G}_5^k(a_1, b_1, \dots, B_1, y) = \mathbf{G}_5(a_k, b_k, \dots, B_k, y), \quad k \in \mathbb{Z},$$

where

$$\begin{aligned} a_k &= \frac{1}{12}(-(\sqrt{6} + 2i))^k + \frac{1}{4}(2 + i\sqrt{2})^k + \frac{1}{3}i^k + \frac{1}{4}(2 - i\sqrt{2})^k \\ &\quad + \frac{1}{12}(\sqrt{6} - 2i)^k, \\ e^{i\pi/4}b_k &= -\frac{\sqrt{3}}{12}(-(\sqrt{6} + 2i))^k + \frac{1}{4}(2 + i\sqrt{2})^k - \frac{1}{4}(2 - i\sqrt{2})^k \\ &\quad + \frac{\sqrt{3}}{12}(\sqrt{6} - 2i)^k, \\ i c_k &= \frac{1}{6}(-(\sqrt{6} + 2i))^k - \frac{1}{3}i^k + \frac{1}{6}(\sqrt{6} - 2i)^k, \end{aligned}$$

$$\begin{aligned}
e^{-i\pi/4} e_k &= -\frac{\sqrt{3}}{12} (-(\sqrt{6}+2i))^k - \frac{1}{4} (2+i\sqrt{2})^k + \frac{1}{4} (2-i\sqrt{2})^k \\
&\quad + \frac{\sqrt{3}}{12} (\sqrt{6}-2i)^k, \\
g_k &= -\frac{1}{12} (-(\sqrt{6}+2i))^k + \frac{1}{4} (2+i\sqrt{2})^k - \frac{1}{3} i^k + \frac{1}{4} (2-i\sqrt{2})^k \\
&\quad - \frac{1}{12} (\sqrt{6}-2i)^k, \\
2e^{i\pi/4} d_k &= (2+i\sqrt{2})^k - (2-i\sqrt{2})^k, \\
4if_k &= (-(\sqrt{6}+2i))^k - (2+i\sqrt{2})^k - (2-i\sqrt{2})^k + (\sqrt{6}-2i)^k, \\
4A_k &= (-(\sqrt{6}+2i))^k + (2+i\sqrt{2})^k + (2-i\sqrt{2})^k + (\sqrt{6}-2i)^k, \\
3B_k &= (-(\sqrt{6}+2i))^k + i^k + (\sqrt{6}-2i)^k.
\end{aligned}$$

For example, for $k = -1$ we get

$$\mathbf{G}_5^{-1}(a_1, \dots, B_1, y) = \begin{bmatrix} \frac{1}{6} - \frac{3i}{10} & \frac{2}{15} - \frac{i}{30} & -\frac{2}{5} & -\frac{2}{15} - \frac{i}{30} & \frac{1}{6} + \frac{3i}{10} \\ -\frac{1}{30} - \frac{2i}{15} & \frac{1}{6} + \frac{i}{10} & \frac{1}{10} + \frac{i}{10} & -\frac{1}{10} - \frac{i}{6} & -\frac{2}{15} - \frac{i}{30} \\ & \frac{2}{5} & \frac{1}{10} - \frac{i}{10} & -\frac{i}{5} & \frac{1}{10} + \frac{i}{10} & -\frac{2}{5} \\ \frac{1}{30} - \frac{2i}{15} & \frac{1}{10} + \frac{i}{6} & \frac{1}{10} - \frac{i}{10} & \frac{1}{6} + \frac{i}{10} & \frac{2}{15} - \frac{i}{30} \\ \frac{1}{6} + \frac{3i}{10} & \frac{1}{30} - \frac{2i}{15} & \frac{2}{5} & -\frac{1}{30} - \frac{2i}{15} & \frac{1}{6} - \frac{3i}{10} \end{bmatrix}.$$

4 Final Remarks

1° It's some natural problem how to defined some power series of $\mathbf{G}_5$. For example, how $e^{\mathbf{G}_5}$ should be defined? By Lemma 1 it can be defined in the following way:

$$\begin{aligned}
\exp(\mathbf{G}_5(a, b, c, d, e, f, g, A, B, y)) &\approx \\
&\approx \sum_{k=0}^N \frac{1}{k!} \mathbf{G}_5(a_k, b_k, c_k, d_k, e_k, f_k, g_k, A_k, B_k, y) = \\
&= \mathbf{G}_5\left(\sum_{k=0}^N \frac{1}{k!} a_k, \sum_{k=0}^N \frac{1}{k!} b_k, \dots, \sum_{k=0}^N \frac{1}{k!} B_k, y\right),
\end{aligned}$$

where N can be chosen arbitrary with a given accuracy, because of the Binet's formulae (12)–(16). On the other hand, we have the following exact formula:

$$\exp(\mathbf{G}_5(a, b, c, d, e, f, g, A, B, y)) = \mathbf{G}_5(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}, \hat{g}, \hat{A}, \hat{B}, y),$$

where

$$\begin{bmatrix} \widehat{a} \\ y^{1/2} \widehat{b} \\ y \widehat{c} \\ y^{3/2} \widehat{e} \\ y^2 \widehat{g} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & \frac{1}{12} \\ \frac{-1}{4\sqrt{3}} & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4\sqrt{3}} \\ \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} \\ \frac{-1}{4\sqrt{3}} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4\sqrt{3}} \\ \frac{1}{12} & -\frac{1}{4} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} e^t \\ e^u \\ e^v \\ e^w \\ e^z \end{bmatrix},$$

for the definition of the parameters t, u, v, w, z see formulae (25)–(29), and

$$\begin{aligned} \widehat{d} &= \widehat{b} + y \widehat{c}, \\ \widehat{f} &= \widehat{c} + y \widehat{g}, \\ \widehat{A} &= \widehat{a} + y \widehat{c}, \\ \widehat{B} &= \widehat{a} + y \widehat{c} + y^2 \widehat{g}. \end{aligned}$$

In the connection with these deliberations it is intriguing in some sense that *Mathematica* have some problem with symbolic Jordan decomposition of the matrix $\mathbf{G}_5(a_1, b_1, \dots, B_1, y)$ from our example.

2° Why are only the matrices 5×5 discussed in the paper? It turned out that generalizations of the case 3×3 (from [8]) onto 4×4 and 5×5 matrices are unique in a certain sense. The case of the 6×6 matrix:

$$\mathbf{G}_6 := \begin{bmatrix} a & yb & y^2c & y^3e & y^4g & y^5k \\ b & A & yd & y^2f & y^3h & y^4g \\ c & d & B & yd & y^2f & y^3e \\ e & f & d & B & yd & y^2c \\ g & h & f & d & A & yb \\ k & g & e & c & b & a \end{bmatrix}$$

with respective recurrence relations for its elements don't work. In consequence the generalization $n \times n$, $n \geq 6$, are not so strong (it is necessary to use less number of parameters or others recurrence relations).

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