A Self-governing Hybrid Model for Noise Removal

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Abstract. Denoising methods based on using fourth-order partial differential equations (PDEs) are providing a good combination of the noise smoothing and the edge preservation without creating blocky effects on the smooth regions of the image. However, finding an optimal choice of model parameters for numerical solver of these techniques is a challenging problem and generally, these model parameters are image-content dependent. In this paper, a hybrid fourth-order PDE-based filter is proposed so that it does not need a manual adjustment of the model parameters. It is shown that by setting the numerical solver of proposed filter for operation at a minor time step-size derived under a data-independent stability condition, the filter can still provide a significantly fast convergence rate. Therefore, the model parameters are reduced to one parameter estimated by using a well-studied mechanism applying in the second-order nonlinear diffusion denoising techniques. Simulation results show that the proposed method can provide a denoised image with higher quality in comparison with that of the existing methods.

Keywords: Denoising, Diffusion, Laplacian, Gradient, Convergence.

1 Introduction

In the last two decades, the use of partial differential equations (PDEs) in image processing has significantly grown. This paper is mainly focused on a class of PDE-based denoising methods known as nonlinear/anisotropic diffusion denoising techniques. The first kind of these methods is introduced by Perona and Malik [1] in 1990 in which the denoised image is the solution of a nonlinear second-order PDE. Since then, there has been a great deal of research in this field which led to the introduction of a variety of nonlinear diffusion denoising techniques (see [2], [3] as a few examples). Although the method proposed by Perona and Malik and its variants are known as good edge preservation denoising methods, these methods tend to produce blocky effects in the images [4].

An effective solution to this problem has been introduced by You and Kaveh [4] in which a fourth-order PDE is used for image noise removal. A significant improvement in the ramp edge preservation and a dramatic reduction of blocky effects are the result of using this fourth-order diffusion denoising scheme. However, the fourth-order diffusion dampens high spatial frequency components (i.e.

noise and step edges) much faster than second-order diffusion [5]. This feature might result in the edge distortion during the evolutionary process of the image denoising especially when smoothing strength of the filter for the detected edges is not effectively reduced by a diffusivity function. The diffusivity function in the fourth-order PDE filters is a function of absolute value of Laplacian of the evolved image and it is more sensitive to the noise compared to the diffusivity function of the second- order nonlinear diffusion filters where the diffusivity is a function of the modulus of the gradient of the evolved image. Moreover, the well-established techniques for estimation of the model parameters in second-order nonlinear diffusion filters are not directly applicable in the fourth-order filters. Therefore, fourth-order denoising filters including the recently developed ones such as [5], [6], and [7] are mainly known as a manual or a man-operated noise removal technique. Due to severe nonlinearity of the fourth-order dynamic flow and strong smoothing property of the Laplacian filter embedded in the structure of these filters, finding the optimal parameters setting is very crucial and these optimal parameters are very image-content-dependent. On the other hand, an optimal parameters selection for these filters usually results in a very slow convergence rate as it has been reported in [4] and [5].

In this paper, a new fourth-order PDE-based denoising method is introduced in which the model parameters is reduced to only one parameter that can be estimated. The simulation results show that the proposed method can outperform the other techniques in terms of the quality of the denoised image. The other significance of the proposed method is its fast convergence rate which is been compared with that of the other techniques as well.

2 Theoretical Background

The basic diffusion equation of Perona and Malik [1] is given by

$$\partial u / \partial t = \text{div.} (c(\|\nabla u\|) \nabla u) , \quad (1)$$

where u is image intensity function, $c(\cdot)$ is diffusivity function by which the diffusion coefficient is calculated and t is time. Symbols of div. and $\|\cdot\|$ are used for mathematical notation of Euclidian norm and divergence respectively. The diffusivity function is a positive and none increasing function of $\|\nabla u\|$. One of these diffusivity functions defined by Perona and Malik is given by

$$c(\|\nabla u\|) = k^2 / (k^2 + \|\nabla u\|^2) , \quad (2)$$

where k is the so-called contrast parameter. There are some effective automatic mechanism for estimation of the contrast parameter such as the schemes introduced in [1] and [3].

In [8], You and his colleagues carried out a detailed analysis to show that the solution of (1) is equal to minimization of an energy functional. If the diffusivity function of (2) is used then the energy functional is

$$R(u) = \int_{\Omega} \frac{k^2}{2} \ln (k^2 + \|\nabla u\|^2) dx dy , \quad (3)$$

where Ω is region of support of u . $R(u)$ is minimized when $\|\nabla u\|^2$ is minimum, which leads to a piecewise constant approximation of u . Therefore, formation of staircase artifacts on the ramp edges is unavoidable. In order to resolve this problem, You et al. [4] introduce a new fourth-order PDE-based denoising method in which the denoised image is obtained by minimization of the potential function given by

$$E(u) = \int_{\Omega} f(|\nabla^2 u|) dx dy, \quad (4)$$

where $f'(s) = sc(s)$. Minimization of (4), after using Euler equation, can be solved by the following gradient descent procedure:

$$\partial u / \partial t = -\nabla^2 (c(|\nabla^2 u|) \nabla^2 u), \quad (5)$$

with the noisy image as the initial condition. By the forward Euler approximation of the $\partial u / \partial t$, the numerical solver of (5) is given by

$$\begin{aligned} u^{n+1} &= u^n - dt \times \nabla^2 (c(|\nabla^2 u^n|) \nabla^2 u^n), \\ u^0 &= u_0 \quad \text{and} \quad n = 0, 1, \dots, N, \end{aligned} \quad (6)$$

where n is number of iterations, dt is time step-size and u_0 is a noisy image. This process is an iterative process. In order to protect the edges from over-smoothing, the process needs to be stopped at a certain number of iterations denoted by N .

Apart from a significant advancement in reduction of blocky effect on the denoised image using (5), the optimal parameter setting for the numerical solver of (6) leads to very slow convergence rate especially, when the level of contaminating noise is moderately high. A recently developed technique known as hybrid model of fourth-order PDE [6] tries to address this problem and by using a relaxed median filter [9] tends to improve the quality of denoised image, when the image is highly noisy. The numerical model of this filter is given by

$$u^{n+1} = RM_{\alpha\omega} (u^n - dt \times \nabla^2 (c(|\nabla^2 u^n|) \nabla^2 u^n)), \quad (7)$$

where RM denotes the relaxed median filter with lower bound α and upper bound ω . This filtering process needs a lower number of iterations to give an estimation of denoised image. On the other hand, computational burden per iteration for this filtering scheme is dramatically higher than that of the You et al. filter.

However, both of these techniques are still suffering from lack of parameter estimation mechanisms particularly for the contrast parameter and the time step-size. As mentioned earlier, the performance of these filters strongly depends on the selection of these parameters and unfortunately, the optimal parameters selection is image-content-dependent. In the following section, a new filtering

scheme is introduced by which the contrast parameter is estimated based on the modulus of the gradient of the evolved image.

3 The Proposed Method

3.1 The Proposed Partial Differential Equation

The ability of edge preservation in the fourth order PDE-based denoising method strongly depends on the extent by which the diffusivity function, $c(\cdot)$, can detect the edges and reduce the strength of the smoothness of the filter for these detected edges. However, when the diffusivity function is a function of the absolute value of the Laplacian of the image, the probability of a false edge-detection in Laplacian map of the image is significantly higher comparing to the case that the diffusivity function is a function of the modulus of the gradient. In fact, as much as the order of the derivative of the image is higher, the sensitivity to the noise is higher [10]. Therefore, the new PDE for denoising is introduced in which the diffusivity function is a function of the gradient modulus of the evolved image in form of

$$\begin{aligned} u^{n+1} &= u^n - dt \times \nabla^2 (c(\|\nabla u^n\|) \nabla^2 u^n) , \\ u^0 &= u_0 \quad \text{and} \quad n = 0, 1, \dots, N , \end{aligned} \quad (8)$$

In this case, the optimal contrast parameter, k , in the diffusivity function, $c(\cdot)$, is proportionally related to the noise level [11] and can be estimated by Canny noise-estimation technique [12] as following:

A histogram of the magnitude values of the gradient throughout the image is computed and k is set to the 80%-90% value of its integral at each iteration.

As the results in the next section show, this simple change can lead to significant improvement in the quality of the denoised image, while the fundamental relation between (8) and its associated potential function need to be explored by conducting a further research. However, it can be shown that the proposed method does not introduce any blocky effect on the image (as it is seen in the second-order nonlinear diffusion filter). The results demonstrated in Fig. 1 give a comparison between the performance of the proposed filter and Perona and Malik filter, where the signal shown in Fig. 1-(a) is been filtered for $t=150$. The diffusivity function for the proposed filter and Perona and Malik filter is chosen to be (2) and contrast parameter estimation is based on the Canny noise estimation mechanism mentioned earlier. While the result obtained by Perona and Malik filter shown in Fig. 1-(b) apparently suffers from formation of Blocky effect, the results of the proposed method in Fig. 1-(c) have no sign of formation of staircase artifacts.

The other important aspect of fourth-order PDE-based denoising techniques is the stability of the numerical solver in order to guarantee the convergence of the solution. Setting a small time step-size, dt , in (8), can guarantee the stability of the dynamic flow, however if dt is too small, it results in numerous number of iterations. Finding an optimal time step-size is a challenging task knowing

Fig. 1. Comparing the result of the proposed method with the result obtained by Perona Malik filter in term of formation of artifacts. Fig. 1-(a) is the original image, Fig. 1-(b) is the result of Perona-malik filter ($t=150$) and Fig. 1-(c) is the result obtained by the proposed method ($t=150$).

that this optimal value is normally image-data-dependent. On the other hand, if the diffusivity function can distinguish between noise and edge with high degree of certainty, the convergence rate of the PDE-based denoising method can be dramatically increased. Comparing the diffusion coefficient map of noisy image of "Lena" at the fifth iteration for the You et al. filter with the one that is obtained by proposed method shows that in the proposed method, the wider regions of the image can be detected as smooth regions, therefore noise reduction is significantly faster than that of (6). Thus, finding a constant dt as a data-independent stability condition for (8) (i.e. very smaller than the optimal time step-size) and setting the numerical solver of (8) so that it operates with this time step-size make the proposed filter applicable to a wide category of images without any stability concern and yet deliver a good convergence rate.

Fig. 2. Comparing the map of $c(|\nabla^2 u|)$ in You et al. filter, (a), with the map of $c(\|\nabla u\|^2)$ in the proposed filter, (b), for test image "Lena" degraded by additive white Gaussian noise with standard deviation of 15. The bright regions in the both maps indicate the regions in which a strong diffusion is carried out.

3.2 Data-Independent Stability Condition

The purpose of this subsection is to find a time step-size, dt , for numerical solver of (8) so that the numerical solver is always convergent to the steady state value. In the other word, for stability of (8), it is needed to show that $\partial u / \partial t \rightarrow 0$ when $t \rightarrow \infty$. If $c(\|\nabla u\|^2) \approx 0$ the changes in u is almost zero and $\partial u / \partial t \approx 0$. Therefore, the worst condition for stability is when $c(\|\nabla u\|^2) = 1$. Note that $c(\|\nabla u\|^2)$ is bounded in $(0, 1]$. For $c(\|\nabla u\|^2) = 1$, the numerical solver of (8) can be written in form of

$$u^{n+1} = u^n - dt \times ((L * L) * u^n), \quad (9)$$

where L is a small Laplacian kernel and symbol of $*$ denotes the convolution operation. If u is arranged in column-wise order, the linear system in (9) can be written as a system of state equations given by

$$u^{n+1} = (I + S) \times u^n, \quad (10)$$

where $(I + S)$ is the state matrix of (9) in which S is an sparse matrix representing the convolution of u with the small convolution kernel of $-dt \times (L * L)$ and I is the identity matrix. When L is a standard Laplacian kernel given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (11)$$

then $-dt \times (L * L)$ is

$$-dt \times (L * L) = dt \times \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 \\ -1 & 8 & -20 & 8 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 \end{bmatrix}, \quad (12)$$

which means that S is a sparse matrix that in each row the nonzero matrix values are in the following order:

$$dt \times [-1 \ -2 \ 8 \ -2 \ -1 \ 8 \ -20 \ 8 \ -1 \ -2 \ 8 \ -2 \ -1] \quad (13)$$

while -20 is always in the main diagonal of S .

For stability of state system in (10) the spectral radius, SP , of $(I + S)$ should be less than one and the same value of dt led to stability of (10) results to stability of

$$u^{n+1} = -(I + S) \times u^n, \quad (14)$$

However, the state system in (14) can be written in standard form of Jacobi solver in form of

$$u^{n+1} + ((2I + S) - I) u^n = 0, \quad (15)$$

Stability of the Jacobi solver in (15) can be obtained if matrix $(2I - S)$ is strictly row-wise diagonally dominant (see [13] page 626 for proof). By definition, the matrix A_{ij} is strictly row-wise diagonally dominant if

$$|a_{rr}| > \sum_{j \neq r} |a_{rj}| \text{ for } r = 1, 2, \dots, n. \quad (16)$$

This condition for $(2I - S)$ means that $2 - 20dt > 44dt$ needs to be satisfied. In the other word, when the time step-size is $dt < 0.0313$, the proposed method using L as discrete approximation of Laplacian is data-independently stable.

In general, for using different Laplacian kernels in implementation of (8), with the same presented argument, one can conclude that the data-independent stability condition for dt is

$$dt < 2 \left(\sum |data \text{ mask of } (L * L)| \right)^{-1}. \quad (17)$$

4 Comparative Results

In this section, we are presenting the comparative results of the proposed method with two other Fourth-order PDEs: 1) the equation of (6) introduced by You et al. [4]. 2) the PDE introduced in [6] known as relaxed median regularized filter. The proposed method does not need to be operated manually, however for the other methods, the suggestive parameters are set to maximize the performance of the filters. For You et al. filter time step size is $dt=0.25$ and $k=0.75$ and the parameters of relaxed median regularized filter are chosen to be $dt=0.1$, $k=3$, as they were suggested in [6].

Three test images of "*Pepper*", "*Cameraman*" and "*House*" have been corrupted by white additive Gaussian noise with standard deviation of 15. In Table 1, an objective comparison between the performances of these filters in terms of signal-to-noise ratio (SNR) of the denoised image and their computational complexity are presented.

Table 1. Objective Comparison Results

Test Images	Noisy Image		Denoised Image		
	SNR(dB)	Method	SNR(dB)	Num. of Iter.	CPU/Iter.
<i>Pepper</i>	11.03	Proposed	17.31	15	0.08
		(6)	15.86	3450	0.03
		(7)	15.10	3	0.15
<i>Cameraman</i>	12.37	Proposed	16.89	7	0.08
		(6)	16.61	3288	0.03
		(7)	13.60	2	0.16
<i>House</i>	9.76	Proposed	17.17	35	0.08
		(6)	15.91	3904	0.03
		(7)	15.38	3	0.16

The results show that the proposed method constantly produces the denoised image with higher SNR. It is important to note that the results are obtained at the optimal number of iterations in which the maximum SNR in evolutionary process of the filters are achieved. If the iterative filtering process is continued after the optimal number of iterations, the SNR of denoised image is reduced due to over-smoothness of edges. The other important feature in the proposed method is its fast convergence rate. As it is shown in Fig. 3, for "Pepper" test image, the convergence rate in the proposed method is much higher than filter of You et al. The computational burden of the filters is measured as CPU time of each iterations provided that they are filtering the same image on the same computer. Thus, the total time for filtering process is a multiplication of CPU/iteration by number of iterations. The relaxed median regularized filter converges faster than the proposed method, however the maximum SNR is significantly lower than that of other methods, and the decay rate of SNR due to over-smoothness of the edges is also very fast. Moreover, the higher computational cost of this filter compared to the proposed filter increases the overall process time due to high computational complexity associated with each iteration of this filtering process.

In Fig. 4, the perceptual quality of the denoised image by the proposed method is compared with that of the other methods. In the first row, the whole image and in the second row, a magnified portion of the image are shown. Each pair of the images is labeled from (a)-(e). The first two images (a) and (b) are the noiseless and the noisy images. In Fig. 4-(c), the denoised image by You et al. filter is shown in which formation of some speckle noise is visible. This drawback

Fig. 3. Comparing the convergence rate of the filters for denoising of test image "Pepper"

Fig. 4. Comparing the perceptual quality of the results. The pair of images labaled (a) to (e) are as the following: (a) noiseless image, (b) noisy image, (c) denoise image using (6), (d) denoised image using (7) and (e) the proposed method.

is known and addressed by You et al. in [4] and it is as a result of choosing small value for k in diffusivity function, however this setting of k is necessary to protect the edges from over-smoothing. In the Fig. 4-(d), the denoised image by the relaxed median regularized filter using (7) is shown. This denoised image is blurred and some staircase artifacts on smooth regions of the image are formed. The last image, shown in Fig.4-(e) is the result of the proposed method in which the extent of denoising and edge preservation is noticeably better than that of the other methods.

5 Conclusion

A method for noise removal based on using the fourth-order PDE has been proposed. A theoretical background of the fourth-order denoising methods has been presented with highlighting the major challenges of parameters estimation and slow convergence rate of these techniques. To resolve these drawbacks, in the proposed method, the diffusion coefficient is calculated using a diffusivity function as a function of modulus of the gradient of the evolved image while, in the existing fourth-order PDE-based filters, the diffusivity function is a function of absolute value of Laplacian of the image. The simulation results show that the proposed method can provide a high SNR and a tangible improvement of perceptual quality of the denoised images in comparison with that of the other techniques. The fast convergence obtained by the proposed method pave the road toward its utilization in real time applications.

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