Fast Multivariate Ordinal Type Histogram Matching

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Abstract. When patterns are represented as histograms, the *earth mover's distance*, *EMD* has been considered an excellent metric between two distributions. *EMD* is formulated as the transportation problem which is a hard optimization problem. In similarity based pattern retrieval problems, computing *EMDs* for all histograms in the database against a query histogram would take too long time for users to wait for the output. Hence, the candidate selection technique is presented to speed up the *EMD* based multivariate ordinal type histogram retrieval problem. It guarantees to find all similar histograms while achieving significant speed up. Theoretical relationships between other metrics for multivariate histograms and their usages are presented as well.

1 Introduction

Distance between histograms plays important role in classification, clustering, and retrieval problems [1]. Among numerous metrics, the *Earth Mover's Distance*, or *EMD* in short, has received great attention in computer vision and pattern recognition [2-4]. It is the minimal amount of work that must be performed to transform one distribution into the other by moving “distribution mass” [2]. This concept was called the *Mallows distance* in Statistics earlier [3]. Other conceptually similar metrics includes the *edit distance* [5,6] which appeared in literature simultaneously as *EMD*. Only *univariate* histograms were considered and depending on the measurement types, $\Theta(b)$, $\Theta(b)$, and $O(b^2)$ efficient solutions for computing *nominal*, *ordinal* and *modulo* type histogram *edit distances* were discovered [5-7] where b is the number of bins.

When histograms are *multivariate*, computing the cost minimization can be formulated as a *transportation problem* [2] which is a hard optimization problem. One of the solutions is the *Simplex method* [9] and the efficient algorithm takes $O(b^3 \log b)$ [9]. Consider the *similarity based histogram retrieval problem*, i.e., retrieving similar histograms out of w histograms in the database to a query histogram, q using the *EMD*. A naïve way to solve the problem would be running the *simplex method* or equivalent algorithm w times which takes unbearably long time for users to wait for the output.

For this reason, a fast method which quickly eliminates most histograms from consideration as possible matches is presented. Let a match be a similar histogram h whose $EMD(h, q) \leq t$, a threshold value. Suppose we have another distance measure, d_x such that $d_x(h, q) \leq EMD(h, q)$ where d_x can be computed very fast. Then we can use

d_x to eliminate h from consideration if $d_x(h, q) > t$. This *candidate selection* method guarantees to find all matches while achieving significant speed up. The *candidate selection* technique was applied successfully for univariate ordinal [6] and modulo [10] type histograms. Here, two efficient multivariate histogram metrics whose computational complexities are linear are introduced for the multivariate ordinal type histogram retrieval problem. Theoretical relationships between metrics are presented.

The rest of the paper is organized as follows. In section 2, the *edit distance* between univariate histograms and the *EMD* between 2D cases are reviewed. Section 3 introduces two other metrics which can be computed very fast for multivariate histograms. Section 4 contains use of two efficient metrics in the similarity based histogram retrieval problem. Finally, section 5 concludes this work.

2 Preliminary: Ordinal Type Histograms

Fig. 1. Univariate ordinal type histogram and its metric

Fig. 2. 2-dimensional 3x3 ordinal histograms

2.1 Univariate Ordinal Type Histograms

Let X be a set of n elements whose possible values are finite. A histogram $H(X)$ of a set X represents the frequency of each value as shown in Figure 1 (a). The frequency value of the i th bin is denoted as $H_i(X)$, e.g., $H_2(X) = 4$ and $H_1(Y) = 2$ in Figure 1 (a). Let b be the number of bins and if b is strictly ordered and increases linearly, then the histogram is *univariate ordinal* type where values are simple scalar values.

Considering a histogram as a block world, the *edit distance* is the minimum amount of necessary movements to build one source histogram to the other target histogram by moving blocks left or right [7] as illustrated in Figure 1 (b) and the formula is:

$$d_{ord_uni}(H(X), H(Y)) = \sum_{i=1}^b \left| \sum_{j=1}^i H_j(X) - H_j(Y) \right| \quad (1)$$

The *edit distance* is calculated by taking the sum of absolute values of prefix sum of difference for each level. Its computational complexity is $\Theta(b)$. Proofs for the correctness and complexity of the algorithm can be found in [7-9]. Note that the *edit*

distance is one of the special cases of the *EMD* in section 2.2. It can be computed as a *transportation problem* [2] or the *minimum difference of pair assignment problem* (*MDPA*) [7] which can be computed in $O(n^3)$ using the *Hungarian method* [11] as depicted in Figure 1 (c). The minimum amount of necessary movements in histograms is the same as the minimum pair assignments in sets. It is important to conceive the eqn (2) because later arguments can be easily proven by the *MDPA* concept.

$$EMD(H(X), H(Y)) = MDPA(X, Y) \quad (2)$$

2.2 Two Dimensional Case

In higher dimensional histograms, values of elements are vectors rather than scalars. If each axis in the vector is *ordinal* and has a finite value, then the histogram is a *d*-dimensional ordinal type histogram. A 2-D ordinal type histogram has the number of bins, $b = r \times c$ and each axis is ordinal, i.e., $x = (1, \dots, c)$ and $y = (1, \dots, r)$.

Consider 3×3 ordinal type histograms where $n = 10$ in Figure 2 as an illustrative example. One can move blocks in the source histogram $H(S)$ to make the target histogram $H(T)$ by moving one block from (1,1) to (1,3), one from (2,1) to (1,3), and one from (3,1) to (2,3). Let every rectangular coordinate bin in $H(S)$ be the source and that in $H(T)$ be the destination. Supply and demand values are bounded (0~ n). Now the metric between two histograms can be formulated as a *standard form of Linear Programming* although it can be shortened as discussed in the later section.

$$\begin{aligned} d_{cm}(H(S), H(T)) &= Z \\ \text{Minimize } Z &= \sum_{i=(1,1)}^{(r,c)} \sum_{j=(1,1)}^{(r,c)} c_{ij} x_{ij} \\ \text{s. t. } \sum_{j=(1,1)}^{(r,c)} x_{ij} &= H_i(S), \text{ for } i = (1,1), \dots, (r,c) \\ \sum_{i=(1,1)}^{(r,c)} x_{ij} &= H_j(T), \text{ for } j = (1,1), \dots, (r,c) \\ \text{and } x_{ij} &\geq 0, \quad \text{for all } i \text{ and } j \end{aligned} \quad (3)$$

Let d_{ord_euc} and d_{ord_cbd} be the *EMDs* with the cost matrix c_{ij} using the Euclidean, L_2 and the city block L_1 distances between two grid indices, respectively. Two feasible solutions for $d_{ord_euc}(H(S), H(T)) = 6.48$ and one for $d_{ord_cbd}(H(S), H(T)) = 8$ are given in Figure 3.

Lemma 1: $d_{ord_euc}(H(X), H(Y)) \leq d_{ord_cbd}(H(X), H(Y))$.

Proof: $MDPA_{ord_cbd}(X, Y) = d_{ord_cbd}(H(X), H(Y))$. Let each pair assignment in the $MDPA_{ord_cbd}(X, Y)$ setting be a_i . Then $MDPA_{ord_cbd}(X, Y) = \sum_{i=1}^n L_1(a_i)$. Let $TEMP(X, Y) = \sum_{i=1}^n L_2(a_i)$. Then $TEMP(X, Y) \leq MDPA_{ord_cbd}(X, Y)$ because $L_2 \leq L_1$. $MDPA_{ord_euc}(X, Y) \leq TEMP(X, Y)$ by definition. $MDPA_{ord_euc}(X, Y) \leq MDPA_{ord_cbd}(X, Y)$ ■

Fig. 3. (a) (b) 2 feasible solutions for d_{ord_euc} (c) d_{ord_cbd}

3 Efficient Metrics

This section introduces two metrics with linear time or lower complexities which shall be used to speed up the *EMD* based histogram retrieval.

3.1 Sum of Direct Subtractions (SDS)

Consider the cost matrix such that the diagonal cells have all 0's and all others have 1's. Histograms with such a cost matrix were referred to *nominal* type histograms [7]. *MDPA* for our examples is given in Figure 4 (a). Let d_{sds} , the *sum of direct subtractions*, *SDS* in short, be the *EMD* with such a cost matrix and its equation is:

$$d_{sds}(H(S), H(T)) = n - \sum_{i=1}^b \min(H_i(S), H_i(T)) = \frac{\sum_{i=1}^b |H_i(S) - H_i(T)|}{2} \quad (4)$$

The eqn (4) is often known as the non-intersection. Its computational complexity is linear: $\Theta(b) = \Theta(r \times c)$. The term, $\sum |H(S) - H(T)|$ is the *city block distance*, $L_1(H(S), H(T)) = 2 \times d_{sds}(H(S), H(T))$ [7]. It should be noted that $L_1(H(S), H(T)) \neq d_{ord_cbd}(H(S), H(T))$ although the term 'city block distance' appears in both metrics.

Let D be the direct subtraction of $H(T)$ from $H(S)$. $D = H(S) - H(T)$ contains positive and negative numbers as shown in Figure 4 (b). Let DP and DN be the sets of elements in positive and negative positions in D , respectively. Every element in DP must be assigned to a certain element in DN as shown in Figure 4 (c). Each assignment cost is 1. *SDS* is simply the cardinality of either DP or DN ; $d_{sds}(H(S), H(T)) = |DP| = |DN|$.

Readers might easily notice that the constraints in eqn (3) can be reduced. $H(S)$ and $H(T)$ can be reduced to $H(DP)$ and $H(DN)$, respectively and non-zero cells in $H(DP)$ and $H(DN)$ are only necessary sources and destinations as shown in Figure 4 (d).

$$MDPA(DP, DN) = MDPA(S, T) \quad (5)$$

$$EMD(H(DP), H(DN)) = EMD(H(S), H(T)) \quad (6)$$

EMD may be computed efficiently by the constraint reduction but there is no asymptotical impact on average. It is stated because it simplifies proofs for following lemmas.

Lemma 2: $d_{sds}(H(S), H(T)) \leq d_{ord_euc}(H(S), H(T))$

Proof: Let $a \in DP$ and $b \in DN$. $c_{a,b} = 1$ for $MDPA_{sds}(DP, DN)$ whereas $c_{a,b} \geq 1$ for $MDPA_{ord_euc}(DP, DN)$. Hence, $MDPA_{sds}(DP, DN) \leq MDPA_{ord_euc}(DP, DN)$. Thus $MDPA_{sds}(S, T) \leq MDPA_{ord_euc}(S, T)$. ■

Lemma 3: $d_{sds}(H(S), H(T)) \leq d_{ord_cbd}(H(S), H(T))$.

Proof: the similar argument to that of Lemma 2.

Fig. 4. SDS for nominal type histograms

3.2 Sum of Grid Differences (SGD)

Every vertical or horizontal grid dissects the histogram into two parts as shown in Figure 5 (a). In order to edit a source histogram to a target histogram, at least eqns (7) or eqn (8) amount of movements must pass the horizontal or vertical grid.

$$gd(k, *) = \left| \sum_{i=1}^k H_{(i, *)}(S) - \sum_{i=1}^k H_{(i, *)}(T) \right| \quad (7)$$

$$gd(*, k) = \left| \sum_{i=1}^k H_{(*, i)}(S) - \sum_{i=1}^k H_{(*, i)}(T) \right| \quad (8)$$

Let the *sum of grid differences (SGD)* be the sum of the eqn (7) for every horizontal grid and the eqn (8) for every vertical grid.

$$d_{sgd}(H(S), H(T)) = \sum_{i=1}^{r-1} gd(i, *) + \sum_{i=1}^{c-1} gd(*, i) \quad (9)$$

It can be very efficiently computed. Let $H^1(X), \dots, H^d(X)$ be projection histograms the histogram is d -dimensional histogram. Note that each H^i is an univariate ordinal type histogram. Thus, SGD can be defined using the eqn (1):

$$d_{sgd}(H(S), H(T)) = \sum_{i=1}^d d_{ord_uni}(H^i(S), H^i(T)) \quad (10)$$

In 2D histograms,

$$= \sum_{i=1}^r |\sum_{j=1}^i H_j^x(S) - H_j^x(T)| + \sum_{i=1}^c |\sum_{j=1}^i H_j^y(S) - H_j^y(T)| \quad (11)$$

where H^x and H^y are vertical and horizontal projection histograms.

Fig. 5. Grid difference distance

Fig. 6. Example for Lemma 4

The computational complexity is clearly $\Theta(d \times b)$ as one must build the projection histograms for each axis. When d is a small constant value like 2 or 3, $\Theta(b)$. Note that in many applications, the projection histograms might be pre-computed before matching. In this case, the complexity for SGD is $\Theta(r + c) \approx \Theta(\sqrt{b})$ for 2D and $\approx \Theta(d \times \sqrt[4]{b})$ for d -dimensional histograms).

Lemma 4: $d_{sgd}(H(S), H(T)) \leq d_{ord_cbd}(H(S), H(T))$

Proof: Let $ag(*, k)$ and $ag(k, *)$ be the number of blocks that actually pass the k th vertical and horizontal grid, respectively when the histogram is edited using the city block distance cost matrix. Then

$$d_{ord_cbd}(H(S), H(T)) = \sum_{i=1}^{r-1} ag(i, *) + \sum_{i=1}^{c-1} ag(*, i) \quad (12)$$

ag cannot be less than gd for every grid. If so, there is no way that the edited source histogram can be the same as the target histogram. However, $ag \geq gd$ because certain blocks can move in different positions on a certain grid in the handshaking manner as shown in Figure 6 (c). ■

Fallacy 1: $d_{sgd}(H(S), H(T)) \leq d_{ord_euc}(H(S), H(T))$.

Proof: a counter example is given in Figure 2, $d_{sgd}(H(S), H(T)) > d_{ord_euc}(H(S), H(T))$, i.e., $8 > 6.48$ while there are other examples like in Figure 6, $d_{sgd}(H(S), H(T)) \leq d_{ord_euc}(H(S), H(T))$, i.e., $0 < 4$ ■

Although the 2D cases are used for the simplicity sake, all claims are valid for any dimensional ordinal type histograms. It should be noted that *SGD* is only for multi-variate *ordinal* type histograms but not for any other type histograms.

4 Multivariate Histogram Retrieval

Consider a query, “Find all similar histograms in a database D to a given histogram q .” This problem can be tackled by defining metrics between histograms, preferably d_{ord_euc} . When a threshold value t is given, the problem can be formalized as:

$$S^t_{ord_euc} = \forall h(d_{ord_euc}(q, h) \leq t \wedge h \in D) \quad (13)$$

A naïve way to solve the problem is to compute the d_{ord_euc} for every histogram $\{h_1, \dots, h_w\}$ where w is the number of histograms in D which takes too long. A better way is using the candidate selection technique. Suppose we first compute the candidate set.

$$C^t_{sds} = \forall h(d_{sds}(q, h) \leq t \wedge h \in D) \quad (14)$$

Then the eqn (13) is equivalent to:

$$S^t_{ord_euc} = \forall h(d_{ord_euc}(q, h) \leq t \wedge h \in C^t_{sds}) \quad (15)$$

Instead of computing the computationally expensive d_{ord_euc} for all h 's in D , one can only compute h 's in the candidate set in the eqn (15) to find matches.

In order to utilize *candidate selection* technique, the following necessary conditions must be met.

- C must be computed fast, i.e., $O(b^3 \log b)$
- C must contain all matches, i.e., $C \supset S^t_{ord_euc}$

The first condition is met as stated in the previous section; computing $d_{sds}(q, \forall h)$ takes $\Theta(wb)$. Theorems of the present section are for the second necessary condition.

Theorem 1: $S^t_{ord_euc} \subset C^t_{sds}$

Proof: Consider an h whose $d_{ord_euc}(q, h) \leq t$. From the Lemma 2, $d_{sds}(q, h) \leq d_{ord_euc}(q, h) \leq t$. Hence, $S^t_{ord_euc} \subset C^t_{sds}$ as shown in Figure 7 (a) ■

Suppose one is looking for all histograms in eqn (16).

$$S^t_{ord_cbd} = \forall h(d_{ord_cbd}(q, h) \leq t \wedge h \in D) \quad (16)$$

Then, the *SDG* can be utilized to produce a candidate set:

$$C^t_{sgd} = \forall h(d_{sgd}(q, h) \leq t \wedge h \in D) \quad (17)$$

Moreover, one can also utilize the *SDS* as another candidate set. Hence the eqn (16) is equivalent to:

$$S^t_{ord_cbd} = \forall h(d_{ord_cbd}(q, h) \leq t \wedge h \in (C^t_{sds} \cap C^t_{sgd})) \quad (18)$$

Figure 7 (b) illustrates the eqn (18).

Fig. 7. Relationship between candidate sets

Theorem 2: $S_{ord_cbd}^t \subset (C_{sds}^t \cap C_{sgd}^t)$

Proof: $S_{ord_cbd}^t \subset C_{sds}^t$ by Lemma 3 and $S_{ord_cbd}^t \subset C_{sgd}^t$ by Lemma 4. ■

Unfortunately as depicted in Figure 7 (c), C_{sgd}^t cannot be a candidate set for $S_{ord_euc}^t$; $S_{ord_euc}^t \not\subset C_{sgd}^t$ because of Fallacy 1. Nevertheless, there is a way to utilize the SGD for d_{ord_euc} . Let p and q be points in d -dimensional *Euclidean space* and let L_1 and L_2 be the *city block* and *Euclidean* distances between two points, respectively.

Axiom 1: $L_2(p, q) \leq L_1(p, q) \leq \sqrt{d} \times L_2(p, q)$.

This bound can be shown by a simple geometry.

Lemma 5: $d_{ord_cbd}(H(X), H(Y)) \leq \sqrt{d} \times d_{ord_euc}(H(X), H(Y))$

Proof: Let each pair assignment in $MDPA_{ord_euc}$ be a_i . $MDPA_{ord_euc}(X, Y) = \sum_{i=1}^n L_2(a_i)$.

Let $TEMP(X, Y) = \sum_{i=1}^n L_1(a_i)$, then $\sum_{i=1}^n L_2(a_i) \leq \sum_{i=1}^n L_1(a_i) \leq \sum_{i=1}^n \sqrt{d} L_2(a_i)$ by Axiom 1. $TEMP(X, Y) \leq \sum_{i=1}^n \sqrt{d} L_2(a_i) = \sqrt{d} \times \sum_{i=1}^n L_2(a_i)$.

$TEMP(X, Y) \leq \sqrt{d} \times MDP A_{ord_euc}(X, Y)$ and $MDPA_{ord_cbd}(X, Y) \leq TEMP(X, Y)$ by definition. $\therefore MDP A_{ord_cbd}(X, Y) \leq \sqrt{d} \times MDP A_{ord_euc}(X, Y)$ ■

$$S_{ord_cbd}^{t \times \sqrt{d}} = \forall h(d_{ord_cbd}(q, h) \leq t \times \sqrt{d} \wedge h \in D) \quad (19)$$

Lemma 6: $S_{ord_cbd}^t \subset S_{ord_euc}^t \subset S_{ord_cbd}^{t \times \sqrt{d}}$

Proof: by Lemma 1 and Lemma 5 ■

$$\text{Let } C_{sgd}^{t \times \sqrt{d}} = \forall h(d_{sgd}(q, h) \leq t \times \sqrt{d} \wedge h \in D) \quad (20)$$

Theorem 3: $S_{ord_euc}^t \subset (C_{sgd}^{t \times \sqrt{d}} \cap C_{sds}^t)$

Proof: $S_{ord_euc}^t \subset S_{ord_cbd}^{t \times \sqrt{d}}$ by Lemma 6. $S_{ord_cbd}^{t \times \sqrt{d}} \subset C_{sgd}^{t \times \sqrt{d}}$ by Theorem 2. ■

In all, computing $S_{ord_euc}^t$ by eqn (13) setting would take too long for users to wait for the output. Some speed up can be achieved in eqn (15) setting. Further speed up can be achieved by the following eqn (21) setting.

$$S_{ord_euc}^t = \forall h(d_{ord_euc}(q, h) \leq t \wedge h \in (C_{sgd}^{t \times \sqrt{d}} \cap C_{sds}^t)) \quad (21)$$

Corollary 1: $\exists h(h \in (C_{sds}^t - C_{sgd}^{t \times \sqrt{2}}))$

Corollary 2: $\exists h(h \in (C_{sgd}^{t \times \sqrt{2}} - C_{sds}^t))$

Proof: Let q , a , and b be 3x3 histograms with $n = 20$.

$$\begin{array}{ccc} 2 & 3 & 2 \\ q = 1 & 2 & 3 \\ 2 & 2 & 3 \end{array} \quad \begin{array}{ccc} 1 & 3 & 2 \\ a = 2 & 1 & 1 \\ 3 & 3 & 4 \end{array} \quad \begin{array}{ccc} 0 & 1 & 2 \\ b = 3 & 3 & 4 \\ 1 & 4 & 2 \end{array}$$

If $t = 4$, a and b are examples of Corollary 1 and 2, respectively. $d_{sds}(q, a) = 4 \leq t$ where $d_{sgd}(q, a) = 6 > t \times \sqrt{2}$. $d_{sds}(q, b) = 6 > t$ where $d_{sgd}(q, b) = 5 \leq t \times \sqrt{2}$. ■

Corollaries 1 and 2 assure that using the eqn (21) is better than using either one of candidate sets alone. In other words, there is no need to compute $d_{ord_euc}(q, a) = 4.8284 > t$ and $d_{ord_euc}(q, b) = 6.4142 > t$ because their values are guaranteed to be larger than t by Theorem 3.

For the experiments, 101 histograms were randomly generated. Using the leave one out method [1], the table 1 shows the average percentage of retrieved similar histograms. For instance of a 3x5 histogram of $n=70$ and $t=15$, only 16.79% of histograms are needed to be validated for matches. The smaller t , the smaller the candidate set size.

Table 1. Average candidate and match set sizes in %

$r \times c$	N	t	$ C_{sds}^t $	$ C_{sgd}^{t \times \sqrt{2}} $	$ C_{sgd}^{t \times \sqrt{2}} \cap C_{sds}^t $	$ S_{ord_euc}^t $
3x5	50	10	7.9%	21.4%	4.65%	0.38%
3x5	50	12	23.9%	29.7%	13.60%	2.75%
3x5	70	12	6.8%	18.3%	3.45%	0.55%
3x5	70	15	27.3%	41.1%	16.79%	3.11%
5x7	100	28	16.2%	30.8%	7.94%	0.12%
5x7	100	30	30.3%	37.6%	15.43%	0.30%

The algorithm using the candidate selection technique for the d_{ord_euc} based d -dimensional ordinal type histogram retrieval problem is:

Input: q , D , and t .
Output: $S_{ord_euc}^t = M$.
Procedure:
1. $M = \emptyset$
2. for $i = 1$ to w
3. if $d_{sgd}(q, h_i) \leq \sqrt{d} \times t$
4. if $d_{sds}(q, h_i) \leq t$
5. if $d_{ord_euc}(q, h_i) \leq t$
6. $h_i \in M$
7. return M

But for lines 3 and 4, it would take too long. The order of these lines matters. If the projection histograms for every histogram in D are pre-computed, $d_{sgd}(q, \forall h)$ takes $\Theta(w \times (r+c))$ much faster than $d_{sds}(q, \forall h)$ which takes $\Theta(w \times (r \times c))$ for 2D histograms and thus SGD should precede SDS . Then the running time complexity for the algorithm is $\Theta(w \times (r+c)) + \Theta(l C_{sgd}^{r \times \sqrt{2}} | \times (r \times c)) + O(l C_{sgd}^{r \times \sqrt{2}} \cap C_{sds}^t | \times b^3 \log b)$. With a little modification, the above algorithm can be used for the *instance based classifier* [1]. However, in most classification problems, the threshold value is not given. One may set $t = \min(d_{sgd}(q, h))$ and slowly increase t to find the nearest one.

5 Conclusion

Similarity based multivariate histogram retrieval problem using the *EMD* as a metric was considered. The candidate selection method which quickly eliminates most histograms from consideration as possible matches was presented. All matches, *i.e.*, similar histograms to a query histogram are still guaranteed to be found while achieving significant speed up. Main contribution of this article is introducing two metrics which satisfy the two necessary conditions to produce a candidate set C for the *EMD* based multivariate histogram retrieval problem.

In addition to aforementioned two necessary conditions for the candidate selection method, a desirable condition is that the expected $|C|$ must be small, *i.e.*, $|C| \ll |D|$. Although Theorems in this paper imply $|C| \leq |D|$, finding the expected $|C|$ theoretically which requires probability analysis is one of the open problems. Finding a new *candidate selection* function such that the expected $|C|$ is small is also a challenging problem, *e.g.*, an easy one might be finding a tighter bound in Lemma 5.

An example of 3D ordinal type histograms is the *RGB* histogram because each component histogram is ordinal type. Consider *HSV* histogram which is a multivariate heterogeneous type histogram. The *Hue* component histogram is *modulo* type whereas *Saturation* and *value* components are ordinal type. Dealing with heterogeneous type multivariate histograms is another open problem.

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