Rank Classification of Linear Line Structure in Determining Trifocal Tensor

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Abstract. The problem we address is: given line correspondences over three views, what is the condition of the line correspondences for the spatial relation of the three associated camera positions to be uniquely recoverable? We tackle the problem from the perspective of trifocal tensor, a quantity that captures the relative positions of the cameras in relation to the three views. We show that the rank of the matrix that leads to the estimation of the tensor reduces to 7, 11, 15 respectively for line pencil, point star, and ruled plane, which are structures that belong to linear line space; and 12, 19, 23 for general ruled surface, general linear congruence, and general linear line complex. These critical structures are quite typical in reality, and thus the findings are important to the validity and stability of practically all algorithms related to structure from motion and projective reconstruction using line correspondences.

1 Introduction

We address the question of, given line correspondences over three views, what is the condition of the observed lines for the relative geometry of the three associated camera positions to be uniquely recoverable in space? There are at least two issues: what particular structure the lines in space belong to, and whether there are enough of the lines. In this work we deal with both the structure issue and the quantity issue, and we deal with them from the perspective of trifocal tensor – the $3 \times 3 \times 3$ quantity that captures the relative positions of the cameras in relation to the views. We investigate whether the observed lines are of a too specific structure in 3-D space to allow the trifocal tensor to be uniquely recoverable, and if they indeed are, by how much they are shy of a general enough structure. We also investigate, for each of the investigated line structures, how many lines are minimally needed to allow the full information of the line structure to be reflected in the image projections.

A set of observed lines in space are called *critical* if there are multiple projectively non-equivalent configurations of the camera positions that can picture the same image triplet of the lines. We shall examine the different classes of structure of the line set that are critical. In all cases, we assume that the camera positions are general with respect to the set of lines, meaning that the images

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are not degenerate images like those having multiple lines projected to the same image line, a result of accidental alignment.

Trifocal tensor, an important quantity in multiple view geometry, is the counterpart of fundamental matrix. Trifocal tensor has drawn much attention in the last decade or so, for it can be computed directly from image correspondences, and its determination would allow camera matrices and in turn projective structure of the imaged scene to be reconstructed. Existing algorithms of determining trifocal tensor are mainly iterative and require initialization typically from a linear method. The degeneracy in using linear method to determine camera motion has been studied in a number of works. However, the works are mainly about the cases where point correspondences are used. There have been a few earlier works on the cases where line correspondences are used, but the treatment has been centered around the use of direct geometric primitives like the relative displacements and orientations of the cameras [1]. Since trifocal tensor has the essence of expressing the relative positions of all cameras in relation to the image observables in a single collective term, we believe the study would be simpler and the degeneracy be more easily quantified if the issue is pursued from the angle of trifocal tensor, and this work aims at doing that.

Line correspondences are in many ways as indispensable as point correspondences. There is evidence [2,3] that 3-D reconstruction from lines could be more accurate than from points. Moreover, linear line structures as the structures to be addressed in this paper do appear quite often in real scenes, as exemplified in Fig. 1. Putting it simply, linear line space includes three subspaces: line pencil as a family of lines that contain the same point and that intersect at a non-incident line (in other words, the lines are both co-planar and co-intersecting at a point on the plane); point-star (also named bundle of lines) as a family of lines that are concurrent at a point; ruled plane (also named field of lines) as a family of lines incident on a particular plane. More general than linear line space, linear line structures include linear line complex which are lines satisfying one linear constraint, linear line congruence satisfying two linear constraints, and linear ruled surface-lines satisfying three linear constraints. In Fig. 1(a), a typical linear line complex structure is shown. The Eiffel tower in Fig. 1(b) can also be approximated as several sets of line bundle and line fields that belong to linear line congruence. The building in Fig. 1(c) consists of a field of lines that are coplanar and they also belong to linear line congruence.

2 Previous Work

In this work we are concerned primarily with recovering camera displacement or camera motion from visual observations. Point and line features can both be used in motion estimation from multiple views. Certain algebraic properties of correspondences over multiple views are summarized in [4] and [5]. Critical configuration of point features, in particular, has already drawn much attention [6,7,8] in the literature, and the studies have been about the condition of the point features under which the camera motion cannot be determined uniquely. A

Fig. 1. Scenes consisting of (left) linear line complex; and linear line congruence ((middle) the sub-class: point-star, and (right) the sub-class: ruled plane) respectively

classification of multiple-view critical configurations is provided in [9], and [10] summarizes the latest results on the subject.

As for line features, several structure and motion estimation algorithms that use line correspondences have been proposed in the literature ([11], [12,13,14]). However, the ambiguity issue possibly involved in the use of line correspondences has not been given as much attention as that to the use of point correspondences. A pioneering work was done by Buchanan [15,16] who pointed out that $(3, 6, 5)$ -congruence is a critical line set, and suggested a few line sets that defeat the Liu-Huang algorithm [17] of recovering motion and structure from line correspondences. Maybank [18] continued Buchanan's work by choosing to use Semple's representation to describe lines so as to let line congruence be parameterized by surfaces of lower degree. However, Semple's representation does not have a direct way of expressing the projection of 3-D lines to image lines. In [19] Navab and Faugeras addressed the analysis of the critical sets of lines in the Euclidean domain using two equations from Liu and Huang [17], and related the degeneracy analysis to the Liu-Huang algorithm. The treatment was however in terms of geometric primitives of the Euclidean space directly related to individual camera positions and orientations, not the collective quantity trifocal tensor. As discussed earlier, trifocal tensor is a concise quantity that captures how the positions and orientations of all cameras are related to image observations. There is value in treating the degeneracy issue in terms of trifocal tensor, and it is especially so when observables are not direct 3-D but only image data. In particular, as we shall see in this paper, the rank of the tensor estimation matrix involved in the trifocal tensor's determination could be a measure of the degeneracy of the observed line set, and the parallel measure in direct geometric primitives is not as accessible.

The work of Stein and Shashua [20] is among the very few that look at the degeneracy of line structures from the viewpoint of trifocal tensor. The work points out that the tensor estimation matrix has the rank reduced to 23 if the observed line set belong to the structure of linear line complex (LLC).

There are however line structures other than LLC in reality. This work aims at providing a comprehensive treatment of the various critical configurations of line structures that can be present in real scenes, and how they impact the

determination of trifocal tensor from image data. The problem is challenging because line correspondences are in many ways less easy to handle than point correspondences, for the reason that their representation, say the Plücker's coordinates, contain nonlinear constraint (Plücker equality) over the coordinates.

3 Preliminaries and Notations

Plücker line coordinates. Given two 3-D points $\mathbf{M}^T \sim (\bar{\mathbf{M}}^T|m)$ and $\mathbf{N}^T \sim (\bar{\mathbf{N}}^T|n)$ of a 3-D line to be represented, the Plücker line coordinates of the line are the homogeneous 6-vector $\mathbf{L} \sim (\mathbf{a}^T|\mathbf{b}^T)^T$, where $\mathbf{a} = \bar{\mathbf{M}} \times \bar{\mathbf{N}}$, $\mathbf{b} = m\bar{\mathbf{N}} - n\bar{\mathbf{M}}$. The Plücker coordinates also satisfy the Plücker equality $\boldsymbol{\Omega} = \mathbf{a}^T \mathbf{b} = 0$. As a general 3-D line has four degrees of freedom, with the Plücker representation, a line in 3-D can be represented as a point in $\mathcal{P}^4$ [18], or a point that lies on a non-singular quadric $\boldsymbol{\Omega}$ in $\mathcal{P}^5$, called the *Klein model* [21].

Camera line projection matrix. Suppose the camera matrix that projects 3-D point to image point is the 3×4 matrix $\mathbf{P} \sim (\bar{\mathbf{P}}|\mathbf{p})$, then the 3×6 line projection matrix [13,14] that projects 3-D line (in the Plücker coordinates as a 6-vector) to image line (in the projective coordinates of $\mathcal{P}^2$ as a 3-vector) is

$$\tilde{\mathbf{P}} \sim [\det(\bar{\mathbf{P}})\bar{\mathbf{P}}^{-T}||_{[\mathbf{p}]_x\bar{\mathbf{P}}}. \quad (1)$$

4 Linear Structure

General 3-D lines form a quadruply infinite system $\mathcal{P}^4$ [22]. Overallly speaking, specific family of lines in the 4-dimensional line space that possess 1 degree of freedom (DoF), 2 DoFs, and 3 DoFs are respectively called ruled surface, line congruence, and line complex in the terminologies of classical line geometry [21]. In particular, if the constraints that reduce the degrees of freedom of the line set are all linear, the above structures are called *linear ruled surface*, *linear line congruence*, and *linear line complex* respectively, and as a collection they are called **linear line structures**. In this work only line structures with linear constraint(s) are investigated; in other words we only consider linear line structures. On top of the above, if the Plücker equality that governs the 6 line coordinates happens to be linear, then the linear line structure is reduced to the **linear line space**, which has linear ruled surface reduced to the sub-structure *line pencil*, and linear line congruence reduced to *point-star* or *ruled plane*. Due to limitation of space, below we only describe linear line congruence and linear line complex in more details for illustration purpose.

4.1 Linear Line Congruence

Line congruence is defined by two independent equations, as the intersection of two complexes. Linear line congruence is line congruence with the two constraints being linear. It is the intersection of two distinct linear line complexes $\mathbf{L}_1, \mathbf{L}_2$. In Plücker coordinates with the norm scaled to be 1, a line $\mathbf{L} \sim (\mathbf{a}^T|\mathbf{b}^T)^T$ in a linear line congruence is bounded by the following: $\mathbf{a}_1^T \mathbf{L} = \mathbf{a}_2^T \mathbf{L} = 0$, $\mathbf{a}^T \mathbf{b} = 0$, $|\mathbf{L}| = 1$, where $\mathbf{a}_1, \mathbf{a}_2$ are constant vectors.

4.2 Linear Line Complex

Linear line complex is constrained by a single equation, leaving only 3 DoFs for any line in the line system. Linear line complex is line complex with the governing constraint being linear. It is the 3-dimensional linear manifold of lines defined by one linear equation. In Plücker coordinates with the norm scaled to be 1, a line $\mathbf{L} \sim (\mathbf{a}^T | \mathbf{b}^T)^T$ in a linear line complex is: $\mathbf{a}_1^T \mathbf{L} = 0$, $\mathbf{a}^T \mathbf{b} = 0$, $|\mathbf{L}| = 1$. If the 6-vector $\mathbf{a}_1$ satisfies the Plücker equality ($\Omega_{\mathbf{a}_1, \mathbf{a}_1} = 0$), meaning that $\mathbf{a}_1$ is the Plücker coordinates of a 3-D line, then all lines of the linear line complex (LLC) meet the line $\mathbf{a}_1$ as its axis, and the LLC is called special or singular. Otherwise, the LLC is called general or regular. The degeneracy study of Stein and Shashua in [20] is about special linear line complex.

5 Critical Configurations of Line Structure in 3-View Analysis

In this work we examine how the estimation of the trifocal tensor from an image triplet would behave should the observed line correspondences over the three views be from particular line systems in 3-D. In particular, we examine how the rank of the tensor estimation matrix $\mathbf{A}$ would be affected, and how distant each of the particular line systems is from the necessary system that allows the trifocal tensor to be uniquely determined.

Line Projection. For any 3-D line $\mathbf{L}$ visible in three given views, its image projections $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ in the three views are related to $\mathbf{L}$ in Plücker coordinates by:

$$\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{l} \\ \mathbf{l}' \\ \mathbf{l}'' \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{P}} \\ \tilde{\mathbf{P}}' \\ \tilde{\mathbf{P}}'' \end{bmatrix} \mathbf{L} \quad (2)$$

where $\tilde{\mathbf{P}}, \tilde{\mathbf{P}}', \tilde{\mathbf{P}}''$ are the three cameras' line projection matrices.

In the above, $\mathbf{L}$ can be taken to include the coordinates of all N 3-D lines visible in the image data. In that case $\mathbf{L}$ will be a $6 \times N$ matrix, $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ will be each a $3 \times N$ matrix, and the entire left side of the above equation, collectively referred to as the $\bar{\mathbf{K}}$ matrix, is about a $9 \times N$ matrix that captures all image line observations in all views of all N lines. As we shall advocate below, this $9 \times N$ observation matrix $\bar{\mathbf{K}}$ is all it takes to define the tensor estimation matrix $\mathbf{A}$ – the key to the determination of the trifocal tensor.

In the rest of paper we also need to access the individual rows of the image line observations $\mathbf{l}$ or $\mathbf{l}'$ or $\mathbf{l}''$ that are visible in View 1, 2, and 3 respectively. We shall define the N -vector $\mathbf{l}_i$ as the i^{th} row of $\mathbf{l}$, and scalar l_{ij} as the j th entry of that row $\mathbf{l}_i$ of $\mathbf{l}$. In other words, $\mathbf{l} = [\mathbf{l}_1^T, \mathbf{l}_2^T, \mathbf{l}_3^T]^T$, and $\mathbf{l}_i = [l_{ij}]$ for all $i = 1, 2, 3$. Similarly, we have N -vector $\mathbf{l}'_i$ and scalar l'_{ij} for image line observations $\mathbf{l}'$ in the second view, and N -vector $\mathbf{l}''_i$ and scalar l''_{ij} for image line observations $\mathbf{l}''$ in the third view.

Tensor Estimation. Each line correspondence over three views gives two independent linear equations for the $3 \times 3 \times 3$ trifocal tensor [5] whose entries can

be listed out in a column fashion as a 27×1 vector $\mathbf{t}$. Given N line correspondences, we have $2N$ linear equations for $\mathbf{t}$, which can be collectively captured by a $2N \times 27$ matrix $\mathbf{A}$. The tensor-equivalent vector $\mathbf{t}$ and the matrix $\mathbf{A}$ which we refer to as the *tensor estimation matrix* or simply *estimation matrix* are related by:

$$\mathbf{A}\mathbf{t} = \mathbf{0}$$

where

$$\mathbf{A} = \begin{bmatrix} \vdots & \vdots \\ \mathbf{0}_{N \times 9} & l_{3i}\mathbf{B}_i & -l_{2i}\mathbf{B}_i \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ -l_{3i}\mathbf{B}_i & \mathbf{0}_{N \times 9} & l_{1i}\mathbf{B}_i \\ \vdots & \vdots & \vdots \end{bmatrix} \quad (3)$$

and $\mathbf{B}_i$ is the i th row ($i = 1, 2, \dots, N$) of the $N \times 9$ matrix $\mathbf{B}$ defined by the image line observations in the following manner:

$$\mathbf{B}_i = [l'_{1i}l''_{1i}, l'_{1i}l''_{2i}, l'_{1i}l''_{3i}, l'_{2i}l''_{1i}, l'_{2i}l''_{2i}, l'_{2i}l''_{3i}, l'_{3i}l''_{1i}, l'_{3i}l''_{2i}, l'_{3i}l''_{3i}] \quad (4)$$

Notice that once the line correspondences are established, $\mathbf{B}$ is known, and so is the estimation matrix $\mathbf{A}$.

The rest of the work will at various places need to employ a rather simple property:

Rank Invariance property. Given any matrix $\mathbf{M} = [\mathbf{m}_i]$, where $\mathbf{m}_i$ is the i -th row of $\mathbf{M}$; then $\text{rank}([a_i\mathbf{m}_i]) \leq \text{rank}(\mathbf{M})$ for all set of scalars $a_i \in \mathcal{R}$.

5.1 Linear Line Space

Linear line space includes only line pencil (with 1 DoF in the line set), point-star (2 DoFs in the line set), and ruled-plane (2 DoFs in the line set).

As captured by Eqn. (2), with N 3-D lines included into the 3-D line set matrix $\mathbf{L}$, the image observations will collectively become a $9 \times N$ matrix $\bar{\mathbf{K}}$. Below we work out the rank property of the estimation matrix $\mathbf{A}$ through the rank property of this observation matrix $\bar{\mathbf{K}}$.

Our presentation will be conducted in this manner. We first show that if the $\bar{\mathbf{K}}$ matrix defined by the line correspondences owns either one of a few rank properties, the rank of the estimation matrix $\mathbf{A}$ will be capped accordingly at a certain value. This we present in three separate lemmas, one for line-pencil, another for point-star, and yet another for ruled plane. We then in a theorem show that the required rank properties of the $\bar{\mathbf{K}}$ matrix are indeed there should the observed line structure be a line-pencil, point-star, and ruled plane respectively.

Due to limitation of space we have to omit some of the proofs in this article.

Lemma 1. For any $9 \times N$ matrix $\mathbf{K} = [\mathbf{k}_1^T, \mathbf{k}_2^T, \mathbf{k}_3^T]^T$, where $\mathbf{k}_i$ is each a $3 \times N$ matrix, suppose

- $\text{rank}(\mathbf{k}_i) = 2$, for $1 \leq i \leq 3$,
- $\text{rank}([\mathbf{k}_i^T, \mathbf{k}_j^T]^T) = 2$, for all i, j such that $1 \leq i, j \leq 3$ and $i \neq j$,
- $\text{rank}(\mathbf{K}) = 2$.

Then if $\mathbf{K}$ is taken as the image line observation matrix $\bar{\mathbf{K}}$ (and thus all $\mathbf{k}_i$, $i = 1, 2, 3$, as $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ respectively) to calculate the estimation matrix $\mathbf{A}$, the rank of $\mathbf{A}$ is at most 7.

Proof. The proof consists of 3 parts: one for the first 9 columns of $\mathbf{A}$, the other for its second 9 columns, and the third for its last 9 columns. In each part, we examine how many of the 9 columns are linearly independent and contribute to the overall rank of $\mathbf{A}$. Let $\mathbf{A}_{*,j}$ be the j th column of the estimation matrix $\mathbf{A}$, for all $j = 1, 2, \dots, 27$.

I First 9 columns of $\mathbf{A}$:

Consider the 1st to 9th columns of $\mathbf{A}$, i.e., the columns $\mathbf{A}_{*,1}$ to $\mathbf{A}_{*,9}$.

- I(a) As pointed out by Eqn. (3), these columns are all about the $N \times 9$ sub-matrix $[-l_{3i}\mathbf{B}_i]$. Both the three sets of columns $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,3}\}$, $\{\mathbf{A}_{*,4}, \mathbf{A}_{*,5}, \mathbf{A}_{*,6}\}$, and $\{\mathbf{A}_{*,7}, \mathbf{A}_{*,8}, \mathbf{A}_{*,9}\}$ are about $[a_i[\mathbf{l}_{1,i}'', \mathbf{l}_{2,i}'', \mathbf{l}_{3,i}'']]$ for some a_i . By the Rank Invariance property stated above, the rank of each of such 3-column sets is no more than $\text{rank}([\mathbf{l}_{1,i}'', \mathbf{l}_{2,i}'', \mathbf{l}_{3,i}'']) = \text{rank}([\mathbf{l}_1'', \mathbf{l}_2'', \mathbf{l}_3'']^T) = \text{rank}(\mathbf{k}_3)$ which is 2 by the first stated rank property of $\mathbf{K}$. In other words, in counting the linearly independent columns, generally we can rule out $\mathbf{A}_{*,3}$, $\mathbf{A}_{*,6}$, and $\mathbf{A}_{*,9}$, and keep only $\mathbf{A}_{*,1}$, $\mathbf{A}_{*,2}$, $\mathbf{A}_{*,4}$, $\mathbf{A}_{*,5}$, $\mathbf{A}_{*,7}$, and $\mathbf{A}_{*,8}$.
- I(b) Similar to the above, by the first stated rank property of $\mathbf{K}$ and the rank invariance property, again it can be deduced that $\text{rank}([\mathbf{A}_{*,j}, \mathbf{A}_{*,j+3}, \mathbf{A}_{*,j+6}]) \leq \text{rank}([\mathbf{l}_1^T, \mathbf{l}_2^T, \mathbf{l}_3^T]^T) = \text{rank}(\mathbf{k}_2) = 2$ for $j = 1, 2, 3$. In other words, $\mathbf{A}_{*,7}$ is dependent of $\mathbf{A}_{*,1}$ and $\mathbf{A}_{*,4}$, and $\mathbf{A}_{*,8}$ is dependent of $\mathbf{A}_{*,2}$ and $\mathbf{A}_{*,5}$.

With the above, within the first 9 columns of $\mathbf{A}$, we have at most 4 columns: $\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}, \mathbf{A}_{*,5}$, that are linearly independent.

- I(c) Next we shall show that $\mathbf{A}_{*,4}$ is linearly independent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}\}$. Observing the second rank property of $\mathbf{K}$, we can pick $\{\mathbf{l}_1, \mathbf{l}_2\}$ as the basis of $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_1', \mathbf{l}_2', \mathbf{l}_3', \mathbf{l}_1'', \mathbf{l}_2'', \mathbf{l}_3''\}$. In other words, we have

$$\begin{aligned} \mathbf{l}_1' &= c_1\mathbf{l}_1 + c_2\mathbf{l}_2 \\ \mathbf{l}_2' &= c_3\mathbf{l}_1 + c_4\mathbf{l}_2 \\ \mathbf{l}_1'' &= c_5\mathbf{l}_1 + c_6\mathbf{l}_2 \\ \mathbf{l}_2'' &= c_7\mathbf{l}_1 + c_8\mathbf{l}_2 \end{aligned} \tag{5}$$

for some $c_i \in \mathbf{R}$.

Suppose that $\mathbf{A}_{*,4}$ is linearly dependent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}\}$. Then there should exist $X, Y \in \mathbf{R}$ such that $\mathbf{A}_{*,4} = X\mathbf{A}_{*,1} + Y\mathbf{A}_{*,2}$ or simply

$$[l_{2i}''] = X[l_{1i}''] + Y[l_{1i}'']$$

where $[*]$ is a vector with $*$ as its column entry. Using Eqn. (5) to expand the above in terms of $\mathbf{l}_{1i}, \mathbf{l}_{2i}$ and equating the coefficients of both sides, the above equation can be reduced to

$$\begin{bmatrix} c_1 c_5 & c_1 c_7 \\ c_1 c_6 + c_2 c_5 & c_1 c_8 + c_2 c_7 \\ c_2 c_6 & c_2 c_8 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} c_3 c_5 \\ c_3 c_6 + c_4 c_5 \\ c_4 c_6 \end{bmatrix}$$

which is an over-constraining set of equations for X, Y , showing that X, Y do not exist in general – a contradiction to the supposition. Thus $\mathbf{A}_{*,4}$ is independent of $\mathbf{A}_{*,1}, \mathbf{A}_{*,2}$.

I(d) By a similar “Proof by Contradiction” step, we can show that $\mathbf{A}_{*,5}$ is linearly dependent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}\}$.

Putting together all the above linear dependencies, $\mathbf{A}_{*,3}, \mathbf{A}_{*,5}, \mathbf{A}_{*,6}, \mathbf{A}_{*,7}, \mathbf{A}_{*,8}, \mathbf{A}_{*,9}$ are linearly dependent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}\}$, and $\text{rank}([\mathbf{A}_{*,1}, \dots, \mathbf{A}_{*,9}]) \leq 3$.

II Second 9 columns of $\mathbf{A}$:

$\{\mathbf{A}_{*,1}, \dots, \mathbf{A}_{*,9}\}$ and $\{\mathbf{A}_{*,10}, \dots, \mathbf{A}_{*,18}\}$ are all about $l_{3i}\mathbf{B}_i$. Thus by the same reason as the first 9 columns of $\mathbf{A}$, we have $\text{rank}([\mathbf{A}_{*,10}, \dots, \mathbf{A}_{*,18}]) = \text{rank}([\mathbf{A}_{*,1}, \dots, \mathbf{A}_{*,9}]) \leq 3$.

$\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}, \mathbf{A}_{*,10}, \mathbf{A}_{*,11}, \mathbf{A}_{*,13}\}$ can be picked as the basis of the first 18 columns of $\mathbf{A}$.

III Last 9 columns of $\mathbf{A}$:

Again by the rank invariance property, the rank of the $2N \times 9$ matrix $\begin{bmatrix} -l_{2i}\mathbf{B}_i \\ l_{1i}\mathbf{B}_i \end{bmatrix}$ is no more than the rank of $[\mathbf{B}_i]$. In other words, the rank property of $\{\mathbf{A}_{*,19}, \dots, \mathbf{A}_{*,27}\}$ is the same as that of $\{\mathbf{A}_{*,1}, \dots, \mathbf{A}_{*,9}\}$ and $\{\mathbf{A}_{*,10}, \dots, \mathbf{A}_{*,18}\}$. Thus, from the proof in Step I, $\mathbf{A}_{*,21}, \mathbf{A}_{*,23}, \dots, \mathbf{A}_{*,27}$ are linearly dependent of $[\mathbf{A}_{*,19}, \mathbf{A}_{*,20}, \mathbf{A}_{*,22}]$, and $\text{rank}([\mathbf{A}_{*,19}, \dots, \mathbf{A}_{*,27}]) \leq 3$.

Finally, we need to determine the linear dependency between the 3 column vectors $\mathbf{A}_{*,19}, \mathbf{A}_{*,20}, \mathbf{A}_{*,22}$ and the basis of $\{\mathbf{A}_{*,1}, \dots, \mathbf{A}_{*,18}\}$. Again, “Proof by contradiction” schemes similar to those above can be used to show that $\mathbf{A}_{*,19}$ is independent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}, \mathbf{A}_{*,10}, \mathbf{A}_{*,11}, \mathbf{A}_{*,13}\}$, but $\{\mathbf{A}_{*,20}, \mathbf{A}_{*,22}\}$ is dependent of $\{\mathbf{A}_{*,1}, \mathbf{A}_{*,2}, \mathbf{A}_{*,4}, \mathbf{A}_{*,10}, \mathbf{A}_{*,11}, \mathbf{A}_{*,13}, \mathbf{A}_{*,19}\}$. Due to page limit, we omit the elaboration of such steps here. To conclude, the rank of $\mathbf{A}$ is at most 7.

Lemma 2. For any $9 \times N$ matrix $\mathbf{K} = [\mathbf{k}_1^T \ \mathbf{k}_2^T \ \mathbf{k}_3^T]^T$, suppose

- $\text{rank}(\mathbf{k}_i) = 2$, for $1 \leq i \leq 3$,
- $\text{rank}([\mathbf{k}_i, \mathbf{k}_j]^T) = 3$, for $1 \leq i, j \leq 3$ and $i \neq j$,
- $\text{rank}(\mathbf{K}) = 3$.

Then if $\mathbf{K}$ is taken as the image line observation matrix $\bar{\mathbf{K}}$ (and thus all $\mathbf{k}_i$, $i = 1, 2, 3$, as $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ respectively) to calculate the estimation matrix $\mathbf{A}$, the rank of $\mathbf{A}$ is at most 11.

Lemma 3. For any $9 \times N$ matrix $\mathbf{K} = [\mathbf{k}_1^T \ \mathbf{k}_2^T \ \mathbf{k}_3^T]^T$, suppose

- $\text{rank}(\mathbf{k}_i) = 3$, for $1 \leq i \leq 3$,
- $\text{rank}([\mathbf{k}_i \ \mathbf{k}_j]^T) = 3$, for $1 \leq i \leq 3, 1 \leq j \leq 3$ and $i \neq j$,
- $\text{rank}(\mathbf{K}) = 3$.

Then if $\mathbf{K}$ is taken as the image line observation matrix $\tilde{\mathbf{K}}$ (and thus all $\mathbf{k}_i$, $i = 1, 2, 3$, as $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ respectively) to calculate the estimation matrix $\mathbf{A}$, the rank of $\mathbf{A}$ is at most 15.

Theorem 1. The rank of tensor estimation matrix is at most 7, 11, 15 for entities in the various structure classes of linear line space: at least 4, 6, 8 lines for line pencil, point-star, and ruled plane respectively.

Proof. Eqn. (2) relates the image observations $[\mathbf{l}^T, \mathbf{l}'^T, \mathbf{l}''^T]^T$ to the 3-D line set $\mathbf{L}$ through the cameras' line projection matrices $\tilde{\mathbf{P}}, \tilde{\mathbf{P}}', \tilde{\mathbf{P}}''$. The relationship leads to the fact that

$$\text{rank}\left(\begin{bmatrix} \mathbf{l} \\ \mathbf{l}' \\ \mathbf{l}'' \end{bmatrix}\right) \leq \min(\text{rank}\left(\begin{bmatrix} \tilde{\mathbf{P}} \\ \tilde{\mathbf{P}}' \\ \tilde{\mathbf{P}}'' \end{bmatrix}\right), \text{rank}(\mathbf{L})) \quad (6)$$

Since $[\tilde{\mathbf{P}}^T, \tilde{\mathbf{P}}'^T, \tilde{\mathbf{P}}''^T]^T$ is of full column rank (=6) in general, we have $\text{rank}([\mathbf{l}^T, \mathbf{l}'^T, \mathbf{l}''^T]^T) = \text{rank}(\mathbf{L})$.

Line Pencil: $\text{rank}(\mathbf{L}) = 6 - 3 - 1 = 2$, with the “3” coming from three linear constraints on $\mathbf{L}$ and the “1” from the Plücker equality (which is degraded to be linear, or else the structure does not belong to linear line space). By the assumption that the projection is general, plus Eqn. (6), it can be deduced that $\text{rank}([\mathbf{l}^T, \mathbf{l}'^T, \mathbf{l}''^T]^T) = 2$. Since $\text{rank}(\mathbf{l}) = \text{rank}(\mathbf{l}') = \text{rank}(\mathbf{l}'') = 2$, we have $\text{rank}([\mathbf{l}^T, \mathbf{l}'^T]^T) = \text{rank}([\mathbf{l}^T, \mathbf{l}''^T]^T) = \text{rank}([\mathbf{l}', \mathbf{l}'']^T) = 2$. All constraints on image lines have thus been considered, and they give rise to the rank properties of $\tilde{\mathbf{K}}$ required by Lemma 1. By Lemma 1, the rank of $\mathbf{A}$ is at most 7 for line pencil. As each line correspondence provides 2 linear constraints on $\mathbf{A}$, the minimum number of line correspondences needed is $\lceil 7/2 \rceil = 4$. Similar proofs can be derived for point star and ruled plane.

In the following subsections we tackle the cases of linear ruled surface, linear line congruence, and linear line complex. We adopt a presentation similar to that for linear line space.

5.2 Linear Ruled Surface

Theorem 2. The rank of tensor estimation matrix is at most 12 if the available 3-view observations are about a set of at least 6 lines that belong to a general linear ruled surface.

5.3 Linear Congruence

Lemma 4. For any $9 \times N$ matrix $\mathbf{K} = [\mathbf{k}_1^T, \mathbf{k}_2^T, \mathbf{k}_3^T]^T$, suppose

- $\text{rank}(\mathbf{k}_i) = 3$, for $1 \leq i \leq 3$,
- $\text{rank}([\mathbf{k}_i, \mathbf{k}_j]^T) = 4$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$ and $i \neq j$,
- $\text{rank}(\mathbf{K}) = 4$.

Then if $\mathbf{K}$ is taken as the image line observation matrix $\bar{\mathbf{K}}$ (and thus all $\mathbf{k}_i$, $i = 1, 2, 3$, as $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ respectively) to calculate the estimation matrix $\mathbf{A}$, the rank of $\mathbf{A}$ is at most 22.

Theorem 3. The rank of tensor estimation matrix is at most 19 if the image observables are about a set of at least 10 lines that belong to a general linear congruence.

Theorem 2 and 3 can be proved with aid from Lemma 3 and 4 respectively.

5.4 Linear Line Complex

Lemma 5. For any $9 \times N$ matrix $\mathbf{K} = [\mathbf{k}_1^T, \mathbf{k}_2^T, \mathbf{k}_3^T]^T$, suppose

- $\text{rank}(\mathbf{k}_i) = 3$, for $1 \leq i \leq 3$,
- $\text{rank}([\mathbf{k}_i, \mathbf{k}_j]^T) = 5$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$ and $i \neq j$,
- $\text{rank}(\mathbf{K}) = 5$.

Then if $\mathbf{K}$ is taken as the image line observation matrix $\bar{\mathbf{K}}$ (and thus all $\mathbf{k}_i$, $i = 1, 2, 3$, as $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ respectively) to calculate the estimation matrix $\mathbf{A}$, the rank of $\mathbf{A}$ is at most 26.

Theorem 4. The rank of tensor estimation matrix is at most 23 if the image observables are about a set of at least 12 lines that belong to a general linear line complex.

Proof. For linear line complex, $\text{rank}(\mathbf{L}) = 6 - 1 = 5$, with 1 coming from one linear constraint. From Eqn. (6), since projection is assumed to be general, $\text{rank}([\mathbf{l}^T, \mathbf{l}'^T, \mathbf{l}''^T]^T) = 5$. From $\text{rank}(\mathbf{l}) = \text{rank}(\mathbf{l}') = \text{rank}(\mathbf{l}'') = 3$, by Lemma 5, it can be concluded that the rank of $\mathbf{A}$ is at most 26 for linear line complex.

However, the above has not yet used the Plücker equality. Even though the Plücker equality is generally nonlinear here, it can be shown that by projecting $\mathbf{L}$ to images 1 and 2, 1 and 3, 2 and 3 respectively, the Plücker equality provides three additional independent linear equations on $\{l_{in}l'_{jn}l''_{kn}\}$, $i, j, k = 1, 2, 3$, $i \neq j \neq k$, $1 \leq n \leq N$. Thus, three DoFs have to be deducted and the rank of $\mathbf{A}$ is at most $26 - 3 = 23$. As each line correspondence provides 2 linear constraints on T , the minimum number of line correspondences needed is $\lceil 23/2 \rceil = 12$. $\square$

An alternative proof has been presented by Stein and Shashua in [20]. The proof here is of a more general nature because it applies not only to special linear line complex but general linear line complex.

6 Conclusion and Future Work

We have supplied a classification of the geometry of a set of lines in space in regard to the determination of trifocal tensor from three views of the lines. In particular, we have presented what rank the estimation matrix would have if the line set is a sub-class (linear ruled surface, linear line congruence, or linear line complex) of linear line structure, or more specifically a sub-structure (point-pencil, point-star, or ruled plane) of linear line space. Trifocal tensor determination is crucial to motion estimation and projective reconstruction, and ambiguity would be present if a less than general set of lines are unfortunately all that can be observed. Notice that in this work not only is the presence of ambiguity pointed out, the extent of the ambiguity is also specified. Such a study over a list of major line structures is unprecedented.

The whole concept can also be looked at from the opposite angle. By examining the rank property of the tensor estimation matrix, one can have a glimpse of what possible line structure the cameras are indeed observing even if it is an under-ranked case. And this can be done without even attempting motion estimation, 3-D structure recovery, or trifocal tensor determination.

We have conducted thorough synthetic and real image data experiments, and they all confirmed the validity of the findings.

Possible future work includes extension of the rank property study for non-linear line structures, and even different camera motions.

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