Calculating the Number of Tunnels

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Abstract. This paper considers 2-regions of grid cubes and proposes an algorithm for calculating the number of tunnels of such a region. The graph-theoretical algorithm proceeds layer by layer; a proof of its correctness is provided, and its time complexity is also given.

Keywords: Number of tunnels, digital topology, 3D grid.

1 Introduction

Calculations of numbers of tunnels (i.e., of the first Betti number β_1) have been a subject of interest in 3D digital topology in the context of image analysis. The Euler characteristic $\chi(K)$ of a bounded 3D set K is given by the Poincaré formula as follows: $\chi(K) = \beta_0 - \beta_1 + \beta_2$, for Betti numbers β_0 (number of connected components), β_1 (see above), and β_2 (number of closed surfaces). For a review on different algorithms for calculating the Euler number, and variations in selecting the adjacency model, see [7]. See also Section 5.3.3 in [6].

Obviously, a known Euler number does not yet specify numbers β_1 or β_2 , even if $\beta_0 = 1$ can be assumed. [3] proposed an incremental, theoretically efficient algorithm for calculating Betti numbers. However, now for more than 12 years, no implementation of this algorithm has been documented somewhere on the net. The authors believe that the algorithm, as proposed in this paper for calculating β_1 , is conceptually easier to follow, and report about its implementation.

This paper presents a hierarchical method for computing the number of tunnels of a 3D binary picture (to be precise, of 2-regions of grid cubes). Images are first processed column by column (runs), then layer by layer, and finally those ‘2D graphs’, deduced from individual layers, are merged into a single ‘3D graph’ in order to compute the number of tunnels. The method is based on the idea of utilizing linear skeletons.

The paper starts with a theoretical part which describes how to infer a ‘2-D’ graph from the ‘3-D’ graphs of the layers. This is followed by a discussion of the algorithm, followed by concluding remarks.

2 Definitions

We apply the grid cell model [6], using 0-, 1-, 2-, or 3-cells in 3D space, which are vertices, grid edges, grid squares, or grid cubes, respectively. Two 3-cells

Fig. 1. G_1 is contracted into G_4 ; G_4 can be further contracted into a loop, but not into a single node (without any edge)

(cubes, for short) c_1 and c_2 are *2-adjacent* iff $c_1 \neq c_2$ and the intersection $c_1 \cap c_2$ contains a 2-cell.

Following Section 4.2.2 of [6], the *merging of two adjacent nodes* p and q in an adjacency graph $G = [S, A]$ is defined by replacing p and q with a new node r that is adjacent to every node in $S \setminus \{p, q\}$ to which p or q was originally adjacent. A finite sequence of merging operations is called a *contraction*; see Figure 1. If graph G_1 is contracted into G_2 , then G_1 and G_2 are *homeomorphic*.

The *linear skeleton* (see Section 6.3.3 of [6]) of a set $M \subseteq \mathbb{R}^3$ is defined by continuous contractions. We transform a linear skeleton into a graph by having a node at each of its singular points (see Section 7.1.3), and arcs between singular points define the edges.

Definition 1. Let S be a 2-region of cubes and G a graph. G is a homotopic graph of S iff G is homeomorphic to the linear skeleton of S .

In Figure 2, G_1 and G_2 are both homotopic graphs of S_1 , and G_3 and G_4 are both homotopic graphs of S_2 .

Two relational structures $[S_1, R_1]$ and $[S_2, R_2]$ are called *isomorphic* iff there exists a one-to-one mapping f from S_1 onto S_2 such that pR_1q iff $f(p)R_2f(q)$, for all $p, q \in S_1$. f is called an *isomorphism*.

Definition 2. Let S be a 2-region and G a homotopic graph of S . S is called a *minimal cubical set* iff G is not isomorphic to any homotopic graph of a 2-region S' , where S' is obtained by removing a cube from S .

Fig. 2. Two 2-regions of cubes with homotopic graphs. S_1 does not contain cube c_1 , and S_2 does not contain cubes c_2 and c_3

A *subregion* is a (non-empty) connected subset of a region. Assume a region adjacency graph and a partition of involved regions into subregions; these subregions define a *subregion adjacency graph*.

Definition 3. Let L be a set of cubes and H a homotopic graph of L . Let $\mathcal{R}$ be a partition of L into subregions without including the infinite background component. Let $[\mathcal{R}, \mathcal{A}]$ be the subregion adjacency graph of $\mathcal{R}$. If $[\mathcal{R}, \mathcal{A}]$ is isomorphic to H , then $[\mathcal{R}, \mathcal{A}]$ is called a homotopic subregion adjacency graph of L with respect to H , denoted by $\bar{H}$.

Let f be the isomorphism; for the whole set $V(H)$ (The set of all vertices in H), $f(V(H))$ is denoted by $\bar{V}(H)$. Figure 3 shows a layer L of cubes. H is a homotopic graph of L . Let $\bar{H}$ be a subregion adjacency graph of L , and $\bar{V}$ the vertices of $\bar{H}$. Each vertex $\bar{v} \in \bar{V}$ is a set of cubes.

In the rest of this paper, we denote a graph by $G = [V, E]$, where $V = V(G)$ is the set of nodes and E is the set of edges. If the relation $\mathcal{A}$ (or set E) is obviously defined for a homotopic subregion adjacency graph $\bar{H} = [\mathcal{R}, \mathcal{A}] = [V, E]$ then we denote $\bar{H}$ just by $V(\bar{H})$ or $\bar{V}$.

Let $G = [V, E]$ be a graph and $V_1 \subseteq V$. Subgraph $[V_1, E_1]$ is called the *restriction* of G on V_1 , where $E_1 = \{v_1v_2 : v_1 \in V_1 \wedge v_2 \in V_1\}$. – The following definition is a special case of Definition 2.64 in [5].

Definition 4. Let S be a family of cubical sets in kD space, where $k = 0, 1, 2, 3$. Let X be the set of i -cells contained in S , with $i \leq k$. Let $p, q \in X$ such that p is a j -cell, q is a $(j - 1)$ -cell and p is the unique j -cell in X incident with q , $j \leq k$. Let $X' = X \setminus \{p, q\}$ and $S' = \cup_{\{c \in X'\}} c$. Then S' is called obtained from S via an elementary collapse of p by q .

Fig. 3. A layer of cubes and a homotopic subregion adjacency graph with respect to a homotopic graph of it

Fig. 4. A minimal cubical set and a homotopic graph

3 Basics

In Figure 4, S_2 is obtained from S_1 after removing 6 cubes. S_2 is a minimal cubical set and G_1 is a homotopic graph of S_2 . S_3 is a minimal cubical set and G_2 is a homotopic graph of S_3 .

Theorem 1. *If a cubical 2-region S is minimal such that a given graph G is a homotopic graph of S , then we have $\beta_1(S) = \beta_1(G)$.*

Corollary 1. *If S is a 2-connected cubical set and G a homotopic graph of S . Then the first Betti number of S is equal to the Betti number of G .*

Let S be a 2-region of cubes with nonempty intersections with grid layer L . Let S_i be a component of the intersection of S with layer L , H_i a homotopic graph of S_i , $\bar{H}_i$ be a homotopic subregion adjacency graph of S_i with respect to H_i , and let $\bar{u}_i, \bar{v}_i, \bar{w}_i \in \bar{H}_i$, for $i = 1, \dots, i_{max}$.

Now consider a pair (v_1, v_2) , where $v_i \in V(H_i)$ and $\bar{v}_i \in V(\bar{H}_i)$, for $i = 1, 2$. Let c be the number of components of $\bar{v}_1 \cap \bar{v}_2$ (It is a set of 2-cells.).

Operation 1. If $c \geq 1$ then add c edges between v_1 and v_2 to graph $H_1 \cup H_2$. Let E be the set of all edges added. Let V_i be the set of such $v_i \in V(H_i)$. In other words, $V_1 = \{v_1 : v_1 \in V(H_1) \wedge v_1 v_2 \in E\}$ and $V_2 = \{v_2 : v_2 \in V(H_2) \wedge v_1 v_2 \in E\}$. Then $V_i \subseteq V(H_i)$, for $i = 1, 2$.

Operation 2. Let $\bar{U}_i$ be the restriction of $\bar{H}_i$ on $\bar{V}_i$, for $i = 1, 2$. For each pair $(\bar{u}_i, \bar{w}_i)$, where $\bar{u}_i, \bar{w}_i \in \bar{U}_i$, let g be the number of 1-paths of 2-cells $(p_0, p_1, \dots, p_n)$ such that p_0 is contained in $\bar{U}_1 \cap \bar{U}_2$ and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$. See Figure 5 for an example.

(Note that $\bar{u}_1 \cap \bar{u}_2$ and $\bar{w}_1 \cap \bar{w}_2$ are also sets of 2-cells.) Delete g edges between u_1 and w_1 in H_1 .

Let $(H_1 \cup H_2)$ be the resulting graph after applying Operations 1 and 2 on $H_1 \cup H_2$. Then we have

Lemma 1. *$(H_1 \cup H_2)$ is a homotopic graph of the set $S_1 \cup S_2$ of cubes.*

Fig. 5. Example with $g = 3$

Let $H = \cup_{i=1}^l H_i$. Repeat Operations 1 and 2 on $H_i \cup H_{i+1}$ until Stop (this is guaranteed to happen, after a finite number of steps), where $i = 1, 2, \dots, l - 1$. Let the resulting graph be (H) . Then, by Lemma 1, we have the following:

Theorem 2. (H) is a homotopic graph of S .

The following algorithm is justified by this theorem.

4 Subprocess and Algorithm

At first we present an algorithm for producing a homotopic subregion adjacency graph and a homotopic graph, for any given layer of cubes within the given 2-region. This is then used to describe our main algorithm.

4.1 Preparation: Runs

A layer of cubes can be decomposed into a number of columns (of cubes); cubes in the same column have the same x -coordinate. A column of cubes can then be decomposed into a number of runs (i.e., *runs*) of cubes. Cubes in the same run have consecutive y -coordinates (i.e., each run is a simple cube-arc). Each run can be represented by its two endcubes.

A run I_1 is *left-(right-)adjacent* to a run I_2 if the x -coordinate of I_1 is less (greater) than that of I_2 , and there exists a cube $c_i \in I_i$ such that $c_1 \cap c_2$ is a 1-cell.

Each left- or right-adjacent run of run I is called a *neighbor* of I . A *run-path* is a sequence of runs $(i_0, i_1, \dots, i_n)$ such that i_{j+1} is the unique right neighbor of i_j ($0 \leq j \leq n - 1$ and $n \geq 1$); i_0 and i_n are called the *endruns* of the path. The endrun with smaller (larger) x -coordinate is called *the first (second) endrun*. A *maximal* run-path is a run-path such that the number of left neighbors of its

first endrun is not one, and the number of right neighbors of its second endrun is also not one. If the number of the neighbors of the second endrun of a run-path p_I is greater than one, then it is called the *head-run* of p_I , denoted by h_{p_I} . Otherwise the first endrun of p_I is called the head-run of p_I . In both cases, we say p_I belongs to h_{p_I} . The second endrun of p_I , different to h_{p_I} , is called the *tail-run* of p_I . We remove the tail-run from the run-path if the number of its neighbors is greater than one. The resulting run-path is called the *reduced* maximal run-path. Thus, if the number of neighbors of the tail-run is one, then the reduced maximal run-path of a run-path coincides with itself.

In Figure 3, the run in Column 18 (the cubes in this run have x -coordinate 18 and y -coordinates between 1 and 20) has 3 left neighbors and 4 right neighbors. There are two maximal run-paths from Column 3 to Column 18. The run in Column 3 is the first end- and tail-run of both, and the run in Column 18 is the second end- and the head-run of both. With the tail-run removed, they induce two reduced maximal run-paths from Column 4 to Column 18. There is also a maximal run-path from Column 5 to 11 (i.e., a disconnected “island” in the largest of the three holes). The run in column 5 is the first end- and head-run of it, and the run in Column 11 is the second end- and tail-run of it. It coincides with its reduced maximal run-path (The tail-run in Column 11 is not removed because it has only one neighbor).

4.2 Single-Layer Subprocess

For a single layer of cubes, do the following:

1. Get the set of all runs, denoted by S_I .
2. For each run in S_I , get and set its left and right neighbors.
3. Get the set of all maximal run-paths, denoted by S_P .
4. For each run-path in S_P , get and set its head- and tail-runs.
5. Get the set of all reduced maximal run-paths from S_P , denoted by S_{RP} .
6. Get the set of all runs, denoted by S_H such that for each run h in S_H , the number of neighbors of h is greater than two, or the number of left neighbors of h equals two and the number of right neighbors of h equals zero, or the number of left neighbors of h equals zero and the number of right neighbors of h equals two.
7. Get the set of head-runs of all run-paths in S_{RP} , denoted by S_{PH} .
8. For each h in $S_H \cup S_{PH}$, reset its neighbors as follows:
 - 8.1. For each pair h_1, h_2 in $S_H \cup S_{PH}$, if they have a common neighbor I , remove I in the set of neighbors of h_i where the index i satisfies $x_i < x_{i+1} \pmod{2}$, where x_i is the x -coordinate of h_i , for $i = 1, 2$.
 - 8.2. For each h in S_H , remove its neighbor if this neighbor is contained in a run-path in S_{RP} (Note that 8.2 and 8.1 are independent of each other because, by definition, a run-path has at least two runs).
9. For each head-run h in $S_H \cup S_{PH}$, let $C_h = C \cup C_N \cup C_p$, where C is the set of all cubes contained in h , C_N is the set of all cubes contained in neighbors of h , and C_p is the set of all cubes contained in reduced maximal run-paths belonging to h (i.e., $C_P = \{c : c \in p \in S_{RP} \wedge p \text{ belongs to } h\}$).

10. Get a homotopic subregion adjacency graph $\bar{H}$ by $S_H \cup S_{PH}$, where each subregion is a set C_h as obtained in Step 9;
11. Get a homotopic graph H from $\bar{H}$.

We use now this single-layer subprocess for describing the main procedure, which computes the number of tunnels of a 2-region of cubes.

4.3 Algorithm

Assume a 2-region S of cubes. We proceed as follows:

1. Decompose S into layers, defined by all those z -coordinates with cells in S . (Each *layer* is thus the subset of S defined by an identical z -coordinate.) Let L_i be the i -th layer of S , where $i = 1, 2, \dots, m$, and m is the number of layers of S .

2. By the single-layer subprocess, construct a homotopic graph H_i for each layer L_i , and a homotopic subregion adjacency graph, $\bar{H}_i$, with respect to H_i .

3. Without loss generality, let L_1 and L_2 be any two consecutive layers. Let H_1 or H_2 be homotopic graphs of L_1 or L_2 , respectively. Let $\bar{H}_1$ or $\bar{H}_2$ be homotopic subregion adjacency graphs with respect to H_1 or H_2 , respectively. For each pair (v_1, v_2) , where $v_1 \in V(H_1)$, $v_2 \in V(H_2)$, $\bar{v}_1 \in V(\bar{H}_1)$, and $\bar{v}_2 \in V(\bar{H}_2)$, do the following:

- 3.1. Apply the FILL algorithm (see, e.g., [6]) to find the number of components of $\bar{v}_1 \cap \bar{v}_2$ (which is a set of 2-cells), denoted by c .

- 3.2. If $c \geq 1$, then add c edges between v_1 and v_2 in graph $H_1 \cup H_2$. Let E be the set of all edges added. Let V_i be the set of these $v_i \in V(H_i)$. In other words, $V_1 = \{v_1 : v_1 \in V(H_1) \wedge v_1 v_2 \in E\}$ and $V_2 = \{v_2 : v_2 \in V(H_2) \wedge v_1 v_2 \in E\}$. Then $V_i \subseteq V(H_i)$, for $i = 1, 2$.

- 3.3. Let $\bar{U}_i$ be the restriction of $\bar{H}_i$ on $\bar{V}_i$, where $i = 1, 2$. For each pair $(\bar{u}_i, \bar{w}_i)$, where $\bar{u}_i, \bar{w}_i \in \bar{U}_i$, apply the FILL algorithm to find the number of 1-paths of 2-cells $(p_0, p_1, \dots, p_n)$ such that p_0 is contained in $\bar{u}_1 \cap \bar{u}_2$, and p_n is contained in $\bar{w}_1 \cap \bar{w}_2$ (where both $\bar{u}_1 \cap \bar{u}_2$ and $\bar{w}_1 \cap \bar{w}_2$ are again sets of 2-cells), denoted by p . Delete p edges between u_1 and w_1 in H_1 .

4. Apply Step 3 to every pair L_i and L_{i+1} of two consecutive layers, for $i = 1, 2, \dots, i_{max} - 1$. Let G be the resulting graph. Then the number of tunnels of S equals $E(G) - V(G) + 1$. For this final step we use a basic result in combinatorial graph theory [1]: Let G be a graph with n nodes, m edges and k connected components. Then we have that $\beta_1(G) = m - n + k$.

4.4 Computational Complexity

Let l be the number of cubes in the considered layer L , and r or c the number of rows or columns in L , respectively.

Step 1 can be computed in time $\mathcal{O}(l)$. Step 2 can be computed in $\mathcal{O}(lr)$. Step 3 can be done in $\mathcal{O}(lc)$. Analogous to Step 1, Steps 4, 5, 6 and 7 can be completed in $\mathcal{O}(l)$. Step 8.1 Step 8.2 can be finished in $\mathcal{O}((lr)^2)$. For Step 9, note that $|S_H \cup S_{PH}| \leq |S_I| \leq l$, $|C| \leq r$, $|C_N| \leq r$, and $|C_P| \leq rc$. That means that Step 9 has

time complexity $\mathcal{O}(lrc)$. Steps 10 and 11 have complexity $\mathcal{O}(l^2)$. In summary, a run of the single-layer subprocess has time complexity $\mathcal{O}((lr)^2)$.

Let S be our input, a 2-region of cubes, and i_{max} the number of layers in S . Let l_i be the number of cubes in the i th layer of S , denoted by L_i . Let r_i or c_i be the numbers of rows or columns in layer L_i , respectively.

Step 1 can be done in $\mathcal{O}(i_{max})$. Step 2 was discussed above. For Step 3.1, the labeling algorithm (as, for example, in [6]) has complexity $\mathcal{O}(r_i c_i)$. Step 3.1 has time complexity $\mathcal{O}(l_i^2 r_i c_i)$. Step 3.2 can be computed in $\mathcal{O}(l_i^2)$. Step 3.3 can be completed in $\mathcal{O}(l_i^2 r_i^2 c_i)$. Step 4 calls the subprocess for all i_{max} layers. Let

$$l = \max_{1 \leq i \leq i_{max}} l_i, \quad r = \max_{1 \leq i \leq i_{max}} r_i, \quad \text{and} \quad c = \max_{1 \leq i \leq i_{max}} c_i$$

The main algorithm has time complexity $\mathcal{O}(i_{max} l^2 r^2 c)$.

5 Concluding Remarks

[2,4] have developed software to compute homology. However, they are (so far) unable to inform about the computational complexity of their algorithms, and so are [5,8]. The latter two compute homology for any finite k -dimensional cubical set (no restrictions on dimensions or shapes). [2] suggests a search for improved algorithms and for new approaches to the task of homology computation while [5] leaves the problem of improving the efficiency of their algorithms to the future. [4] states that time complexity of the proposed algorithm may grow ‘horribly’.

We have presented our algorithm and analyzed its computational complexity, which is relatively low compared to other options. We also believe that the graph-theoretical nature of our algorithm is of general interest.

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