Improvement on the Decay of Crossing Numbers^{*}

Jakub Černý¹, Jan Kynčl¹, and Géza Tóth²

¹ Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
kuba@kam.mff.cuni.cz, kync1@kam.mff.cuni.cz

² Rényi Institute, Hungarian Academy of Sciences, Reáltanoda utca 13-15, H-1053, Budapest, Hungary
geza@renyi.hu

Abstract. We prove that the crossing number of a graph decays in a “continuous fashion” in the following sense. For any $\varepsilon > 0$ there is a $\delta > 0$ such that for n sufficiently large, every graph G with n vertices and $m \geq n^{1+\varepsilon}$ edges has a subgraph G' of at most $(1 - \delta)m$ edges and crossing number at least $(1 - \varepsilon)\text{CR}(G)$. This generalizes the result of J. Fox and Cs. Tóth.

1 Introduction

For any graph G , let $n(G)$ (resp. $m(G)$) denote the number of its vertices (resp. edges). If it is clear from the context, we simply write n and m instead of $n(G)$ and $m(G)$. The crossing number $\text{CR}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. In the optimal drawing of G , crossings are not necessarily distributed uniformly among the edges. Some edges could be more “responsible” for the crossing number than some other edges. For any fixed k , it is not hard to construct a graph G whose crossing number is k , but G has an edge e such that $G \setminus e$ is planar. Richter and Thomassen [RT93] started to investigate the following general problem. We have a graph G , and we want to remove a given number of edges. By *at least* how much does the crossing number decrease? They conjectured that there is a constant c such that every graph G with $\text{CR}(G) = k$ has an edge e with $\text{CR}(G \setminus e) \geq k - c\sqrt{k}$. They only proved that G has an edge with $\text{CR}(G \setminus e) \geq \frac{2}{5}\text{CR}(G) - O(1)$.

Pach, Radoičić, Tardos, and Tóth [PRTT06] proved that for every graph G with $m(G) \geq \frac{103}{16}n(G)$, we have $\text{CR}(G) \geq 0.032\frac{m^3}{n^3}$. It is not hard to see [PT00] that for *any* edge e , we have $\text{CR}(G - e) \geq \text{CR}(G) - m + 1$. These two results imply

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an improvement of Richter–Thomassen bound if $m \geq 8.1n$, and also imply the Richter–Thomassen conjecture for graphs of $\Omega(n^2)$ edges.

J. Fox and Cs. Tóth [FT06] investigated the case where we want to delete a *positive fraction* of the edges.

Theorem A. [FT06] *For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\text{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \text{CR}(G).$$

In this note we generalize Theorem A.

Theorem. *For every $\varepsilon, \gamma > 0$, there is an $n_{\varepsilon, \gamma}$ such that every graph G with $n(G) \geq n_{\varepsilon, \gamma}$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon\gamma}{2394}\right) m(G)$$

and

$$\text{CR}(G') \geq (1 - \gamma) \text{CR}(G).$$

2 Proof of the Theorem

Our proof is based on the argument of Fox and Tóth [FT06], the only new ingredient is Lemma 1.

Definition. Let $r \geq 2, p \geq 1$ be integers. A *2r-earring of size p* is a graph which is a union of an edge uv and p edge-disjoint paths between u and v , each of length at most $2r - 1$. Edge uv is called the *main edge* of the 2r-earring.

Lemma 1. *Let $r \geq 2, p \geq 1$ be integers. There exists n_0 such that every graph G with $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges contains at least $m/3pr$ edge-disjoint 2r-earrings, each of size p .*

Proof. By the result of Alon, Hoory, and Linial [AHL02], for some n_0 , every graph with $n \geq n_0$ vertices and at least $n^{1+1/r}$ edges contains a cycle of length at most $2r$.

Suppose that G has $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges. Take a *maximal* edge-disjoint set $\{E_1, E_2, \dots, E_x\}$ of 2r-earrings, each of size p . Let $E = E_1 \cup E_2 \cup \dots \cup E_x$, the set of all edges of the earrings and let $G' = G \setminus E$. Now let E'_1 be a 2r-earring of G' of maximum size. Note that this size is less than p . Let $G'_1 = G' \setminus E'_1$. Similarly, let E'_2 be a 2r-earring of G'_1 of maximum size and let $G'_2 = G'_1 \setminus E'_2$. Continue analogously, as long as there is a 2r-earring in the

remaining graph. We obtain the $2r$ -earrings $E'_1, E'_2, \dots, E'_y$, and the remaining graph $G'' = G'_y$ does not contain any $2r$ -earring. Let $E' = E'_1 \cup E'_2 \cup \dots \cup E'_y$.

We claim that $y < n^{1+1/r}$. Suppose on the contrary that $y \geq n^{1+1/r}$. Take the main edges of $E'_1, E'_2, \dots, E'_y$. We have at least $n^{1+1/r}$ edges so by the result of Alon, Hoory, and Linial [AHL02] some of them form a cycle C of length at most $2r$. Let i be the smallest index with the property that C contains the main edge of E'_i . Then C , together with E'_i would be a $2r$ -earring of G'_{i-1} of greater size than E'_i , contradicting the maximality of E'_i .

Each of the earrings $E'_1, E'_2, \dots, E'_y$ has at most $(p-1)(2r-1) + 1$ edges so we have $|E'| \leq y(p-1)(2r-1) + y < (2pr-1)n^{1+1/r}$. The remaining graph, G'' does not contain any $2r$ -earring, in particular, it does not contain any cycle of length at most $2r$, since it is a $2r$ -earring of size one. Therefore, by [AHL02], for the number of its edges we have $e(G'') < n^{1+1/r}$.

It follows that the set $E = \{E_1, E_2, \dots, E_x\}$ contains at least $m - 2prn^{1+1/r} \geq \frac{2}{3}m$ edges. Each of $E_1, E_2, \dots, E_x$ has at most $p(2r-1) + 1 \leq 2pr$ edges, therefore, $x \geq m/3pr$. $\square$

Lemma 2. [FT06] *Let G be a graph with n vertices, m edges, and degree sequence $d_1 \leq d_2 \leq \dots \leq d_n$. Let ℓ be the integer such that $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. If n is large enough and $m = \Omega(n \log^2 n)$ then*

$$\text{CR}(G) \geq \frac{1}{65} \sum_{i=1}^{\ell} d_i^2.$$

Proof of the Theorem. Let $\varepsilon, \gamma \in (0, 1)$ be fixed. Choose integers r, p such that $\frac{1}{r} < \varepsilon \leq \frac{2}{r}$ and $\frac{132}{p} < \gamma \leq \frac{133}{p}$. Then there is an $n_{\varepsilon, \gamma}$ with the following properties: (a) $n_{\varepsilon, \gamma} \geq n_0$ from Lemma 1, (b) $(n_{\varepsilon, \gamma})^{1+\varepsilon} > 6pr \cdot (n_{\varepsilon, \gamma})^{1+1/r}$,

Let G be a graph with $n \geq n_{\varepsilon, \gamma}$ vertices and $m \geq n^{1+\varepsilon}$ edges.

Let $v_1, \dots, v_n$ be the vertices of G , of degrees $d_1 \leq d_2 \leq \dots \leq d_n$ and define ℓ as in Lemma 2, that is, $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. Let G_0 be the subgraph of G induced by $v_1, \dots, v_{\ell}$. Observe that G_0 has at least $m/3$ edges. Therefore, by Lemma 1 G_0 contains at least $m/9pr$ edge-disjoint $2r$ -earrings, each of size p .

Let M be the set of the main edges of these $2r$ -earrings. We have $|M| \geq m/9pr \geq \frac{\varepsilon\gamma}{2394}m$. Let $G' = G \setminus M$ and $G'_0 = G_0 \setminus M$.

Take an optimal drawing $D(G')$ of the subgraph $G' \subset G$. We have to draw the missing edges to obtain a drawing of G . Our method is a randomized variation of the embedding method, which has been applied by Leighton [L83], Richter and Thomassen [RT93], Shahrokhi et al. [SSSV97], Székely [S04], and most recently by Fox and Tóth [FT06]. For every missing edge $e_i = u_i v_i \in M \subset G_0$, e_i is the deleted main edge of a $2r$ -earring $E_i \subset G_0$. So there are p vertex-disjoint paths in G_0 from u_i to v_i . For each of these paths, draw a curve from u_i to v_i infinitesimally close to that path. Call these p curves *potential $u_i v_i$ -edges* and call the resulting drawing D .

To get a drawing of G , for each $e_i = u_i v_i \in M$, choose one of the p potential $u_i v_i$ -edges at random, independently and uniformly, with probability $1/p$, and draw the edge $u_i v_i$ as that curve.

There are two types of new crossings in the obtained drawing of G . First category crossings are infinitesimally close to a crossing in $D(G')$, second category crossings are infinitesimally close to a vertex of G_0 in $D(G')$.

The expected number of first category crossings is at most

$$\left(1 + \frac{2}{p} + \frac{1}{p^2}\right) \text{CR}(G') = \left(1 + \frac{1}{p}\right)^2 \text{CR}(G').$$

Indeed, for each edge of G' , there can be at most one new edge drawn next to it, and that is drawn with probability at most $1/p$. Therefore, in the close neighborhood of a crossing in $D(G')$, the expected number of crossings is at most $(1 + \frac{2}{p} + \frac{1}{p^2})$. See figure 1(a).

Fig. 1. The thick edges are edges of G' , the thin edges are the potential edges. Figure shows (a) a neighborhood of a crossing in $D(G')$ and (b) a neighborhood of a vertex v_i in G' .

In order to estimate the expected number of second category crossings, consider the drawing D near a vertex v_i of G_0 . In the neighborhood of vertex v_i we have at most d_i original edges. Since we draw at most one potential edge along each original edge, there can be at most d_i potential edges in the neighborhood. Each potential edge can cross each original edge at most once, and any two potential edges can cross at most twice. See figure 1(b). Therefore, the total number of first category crossings in D in the neighborhood of v_i is at most $2d_i^2$. (This bound can be substantially improved with a more careful argument, see e. g. [FT06], but we do not need anything better here.) To obtain the drawing of G , we keep each of the potential edges with probability $1/p$, so the expected number of crossings in the neighborhood of v_i is at most $\frac{1}{p} 2d_i^2$.

Therefore, the total expected number of crossings in the random drawing of G is at most $(1 + \frac{2}{p} + \frac{1}{p^2})\text{CR}(G') + \frac{2}{p} \sum_{i=1}^{\ell} d_i^2$.

There exists an embedding with at most this many crossings, therefore, by Lemma 2 we have

$$\text{CR}(G) \leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G') + \frac{2}{p} \sum_{i=1}^{\ell} d_i^2 \leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G') + \frac{130}{p} \text{CR}(G).$$

It follows that

$$\left(1 - \frac{130}{p}\right) \text{CR}(G) \leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G')$$

so

$$\left(1 - \frac{130}{p}\right) \left(1 - \frac{1}{p}\right)^2 \text{CR}(G) \leq \text{CR}(G')$$

consequently

$$\text{CR}(G') \geq \left(1 - \frac{132}{p}\right) \text{CR}(G) \geq (1 - \gamma) \text{CR}(G). \quad \square$$

Remark. In the statement of our Theorem we cannot require that *every* subgraph G' with $(1 - \delta)m(G)$ edges has crossing number $\text{CR}(G') \geq (1 - \gamma)\text{CR}(G)$, instead of just *one* such subgraph G' . In fact, the following statement holds.

Proposition. *For every $\varepsilon \in (0, 1)$ there exist graphs G_n with $n(G_n) = \Theta(n)$ vertices and $m(G_n) = \Theta(n^{1+\varepsilon})$ edges with subgraphs $G'_n \subset G_n$ such that*

$$m(G'_n) = (1 - o(1)) m(G_n)$$

and

$$\text{CR}(G'_n) = o(\text{CR}(G_n)).$$

Proof. Roughly speaking, G_n will be the disjoint union of a large graph G'_n with low crossing number and a small graph H_n with large crossing number. More precisely, let $G = G_n$ be a disjoint union of graphs $G' = G'_n$ and $H = H_n$, where G' is a disjoint union of $\Theta(n^{1-\varepsilon})$ complete graphs, each with n^ε vertices and H is a complete graph with $n^{(3+5\varepsilon)/8}$ vertices. We have $m(G) = \Theta(n^{1+\varepsilon})$ and $m(H) = \Theta(n^{(3+5\varepsilon)/4}) = o(m(G))$, since $\frac{3+5\varepsilon}{4} < 1 + \varepsilon$. By the crossing lemma (see e. g. [PRTT06]), $\text{CR}(G) \geq \text{CR}(H) = \Omega(n^{(3+5\varepsilon)/2})$, but $\text{CR}(G') = O(n^{1-\varepsilon} \cdot n^{4\varepsilon}) = O(n^{1+3\varepsilon}) = o(\text{CR}(G))$, because $\frac{3+5\varepsilon}{2} > 1 + 3\varepsilon$. $\square$

On the other hand, we conjecture that we *can* require that a positive fraction of all subgraphs G' of G with $(1 - \delta)m(G)$ edges has crossing number $\text{CR}(G') \geq (1 - \gamma)\text{CR}(G)$. The simplest form of our conjecture is the following.

Conjecture. *For every $\varepsilon > 0$, there is an n_ε and δ such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has the following property. Let G' be a random subgraph of G such that we choose each edge of G independently with probability $p = 1 - \delta$. Then*

$$\Pr[\text{CR}(G') \geq (1 - \varepsilon)\text{CR}(G)] > \delta.$$

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