A Note on Minimum-Area Straight-Line Drawings of Planar Graphs*

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Abstract. Despite a long research effort, finding the minimum area for straight-line grid drawings of planar graphs is still an elusive goal. A long-standing lower bound on the area requirement for straight-line drawings of plane graphs was established in 1984 by Dolev, Leighton, and Trickey, who exhibited a family of graphs, known as *nested triangles graphs*, for which $(2n/3 - 1) \times (2n/3 - 1)$ area is necessary. We show that nested triangles graphs can be drawn in $2n^2/9 + O(n)$ area when the outer face is not given, improving a previous $n^2/3$ area upper bound. Further, we show that $n^2/9 + \Omega(n)$ area is necessary for any planar straight-line drawing of a nested triangles graph. Finally, we deepen our insight into the $4/9n^2 - 4/3n + 1$ lower bound by Dolev, Leighton, and Trickey, which is conjectured to be tight, showing a family of plane graphs requiring more area.

1 Introduction

Area minimization is recognized to be an important aesthetic requirement in Graph Drawing. Besides, drawing planar graphs in the minimum area is a long-standing and fascinating combinatorial problem. In 1984, Dolev *et al.* [5] first exhibited a family of graphs, called *nested triangles graphs*, to show an area lower bound of $(2n/3 - 1) \times (2n/3 - 1)$ for straight-line drawings of plane graphs, that is, graphs with a fixed combinatorial embedding and a fixed outer face. Grids of sizes $(2n - 4) \times (n - 2)$, $(n - 2) \times (n - 2)$, and $\lfloor 2(n - 1)/3 \rfloor \times (4 \lfloor 2(n - 1)/3 \rfloor - 1)$ were shown to be sufficient for straight-line drawings of plane graphs by de Fraysseix, Pach, and Pollack [4], Schnyder [8], and Chrobak and Nakano [3], respectively.

Very little is known when the combinatorial embedding and the outer face of the graphs can be changed. Namely, while area upper bounds for plane graphs trivially extend to planar graphs, there is, as far as we know, no non-trivial lower bound for the area required by planar graphs. In particular, one could ask whether the family of nested triangles graphs, used to show the lower bound for plane graphs, is a good candidate for providing a lower bound in the variable embedding setting. Thus determining the area requirements of nested triangles

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graphs when the outer face can be changed is regarded as one among the most challenging and interesting problems for the Graph Drawing community [2].

In this paper, we show an algorithm to produce straight-line drawings of nested triangles graphs in $(\frac{n}{3} + 3) \times (\frac{2n}{3} + 6) = \frac{2}{9}n^2 + O(n)$ area (Section 3). Such a bound improves the previous best bound of $(\frac{n}{2}) \times (\frac{2n}{3}) = \frac{n^2}{3}$, due to Ossona de Mendez [7]. In the same section we show that any planar straight-line drawing of a nested triangles graph requires $n^2/9 + \Omega(n)$. This result provides, as far as we know, the first non-trivial lower bound for the area required by straight-line drawings of planar graphs in the variable embedding setting. Section 2 contains some background and Section 4 contains our conclusions and some open problems.

2 Preliminaries

A *straight-line grid drawing* of a graph (or *drawing*, for short) is a mapping of each vertex to a distinct point of the plane with integer coordinates and of each edge to a segment between its endpoints. The *bounding box* of a drawing Γ is the smallest rectangle with sides parallel to the axes that completely covers Γ . The *height* (*width*) of Γ is the height (*width*) of its bounding box. The *area* of Γ is the height of Γ multiplied by its width.

A *planar drawing* is such that no two edges intersect except, possibly, at common endpoints. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same circular ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. Two equivalent drawings of the same graph may differ for the outer face. A graph together with a planar embedding and a choice for its outer face is called *plane graph*. A *k-connected* graph G is such that removing any $k - 1$ vertices leaves G connected. A 3-connected (or triconnected) planar graph G admits a unique planar embedding. Hence, different plane graphs can be obtained from G only by choosing different outer faces.

Let t_1 and t_2 be two disjoint 3-cycles of a graph G , and let Γ be a planar drawing of G . We say that t_2 is *nested into* t_1 in Γ if t_2 is drawn in the bounded region of the plane delimited by t_1 . We denote such a relationship by $t_1 > t_2$. A *nested triangles graph* G with n vertices (n is a multiple of 3) is a triconnected graph admitting a planar drawing Γ in which $n/3$ disjoint triangles $t_1, t_2, \dots, t_{n/3}$ can be found such that $t_1 > t_2 > \dots > t_{n/3}$. A nested triangles graph is *maximal* if all its faces are triangles.

Property 1. Let Γ be any planar drawing of a graph G , and let t_1 and t_2 be two disjoint 3-cycles of G such that $t_1 > t_2$ in Γ . The height (width) of t_1 in Γ is at least two units bigger than the height (width) of t_2 .

Based on Property 1, given an n -vertex nested triangles graph G any drawing of G such that $t_1 > t_2 > \dots > t_{n/3}$ requires $(\frac{2n}{3} - 1) \times (\frac{2n}{3} - 1)$ area [5,4]. Such a bound can be achieved with a drawing like the one represented in Fig. 1(a).

Fig. 1. (a) A minimum area drawing of a maximal nested triangles graph. (b) Construction of a three-semi-axes drawing.

Given two integer numbers a and b , the half-line starting at the origin and passing through vertices $(k \cdot a, k \cdot b)$, for any positive integer k , is denoted $\alpha_{(a,b)}$. Let $\alpha_{(a,b)}$, $\alpha_{(c,d)}$, and $\alpha_{(e,f)}$ be three half-lines such that the angle determined by any two consecutive half-lines is less than π . Given an n -vertex nested triangles graph G admitting a planar drawing in which $n/3$ disjoint triangles $t_1, t_2, \dots, t_{n/3}$ can be found such that $t_1 > t_2 > \dots > t_{n/3}$, a *three-semi-axes drawing* Γ of G with axes $\alpha_{(a,b)}$, $\alpha_{(c,d)}$, and $\alpha_{(e,f)}$ is a planar straight-line drawing obtained by suitably placing the vertices of t_i on points $(i \cdot a, i \cdot b)$, $(i \cdot c, i \cdot d)$, $(i \cdot e, i \cdot f)$, with $1 \leq i \leq n/3$. Fig. 1(b) shows an example of construction of a three-semi-axes drawing of a nested triangles graph. Observe that, since from any vertex of triangle t_i a straight-line segment can be drawn to any vertex of triangle t_{i+1} without introducing intersections, a three-semi-axes drawing is always planar.

3 Area Bounds for Nested Triangles Graphs

The following lemma proves the claimed upper bound.

Lemma 1. *Every n -vertex nested triangles graph admits a planar straight-line drawing in $\frac{2}{9}n^2 + O(n)$ area.*

Proof: To prove the statement we restrict to maximal graphs, since any non-maximal nested triangles graph can be augmented to maximal by adding dummy edges. Consider any maximal nested triangles graph G admitting a planar drawing Γ^* in which $n/3$ disjoint triangles $t_1, t_2, \dots, t_{n/3}$ can be found such that $t_1 > t_2 > \dots > t_{n/3}$. We show how to construct a drawing Γ of G in $\frac{2}{9}n^2 + O(n)$ area. Observe that, since G is triconnected, Γ and Γ^* have the same plane embedding (up to a reversal of their adjacency lists) with the exception, possibly, of the choice of the outer face. Also, since n is a multiple of 3, either n is also multiple of 6 (n is even) or n is not multiple of 6 (n is odd).

Suppose n is even. Consider the subgraph G' of G induced by the vertices of $t_{n/6}$ and $t_{n/6+1}$. We label the vertices of $t_{n/6}$ with labels v_1, v_2 , and v_3 , and those of $t_{n/6+1}$ with v_4, v_5 , and v_6 . Such a labeling is based on the degree and the adjacencies of the vertices in G' . Two are the cases, either all vertices of G'

Fig. 2. Cases for the proof of Lemma 1

Fig. 3. A drawing of a nested triangles graph with optimal $\frac{2}{3}n^2 + O(n)$ area

have degree four, or not. In the first case, label the vertices of G' as depicted in Fig. 2(a). In the second case, label the vertices of G' as depicted in Fig. 2(b).

Choose as outer face of Γ the face incident to v_1 , v_2 , and v_5 . Due to the choice of the outer face, Γ contains two sequences S_1 and S_2 of nested triangles, with $S_1 : t_{n/6} > t_{n/6-1} > \dots > t_1$ and $S_2 : t_{n/6+1} > t_{n/6+2} > \dots > t_{n/3}$ (see Fig. 3). Separately construct the drawings of S_1 and S_2 as follows. For S_1 construct a three-semi-axes drawing Γ_{S_1} with axes $\alpha_{(0,1)}$, $\alpha_{(-1,-1)}$, and $\alpha_{(1,-1)}$, such that v_1 , v_2 , and v_3 lie on axes $\alpha_{(0,1)}$, $\alpha_{(-1,-1)}$, and $\alpha_{(1,-1)}$, respectively. Further, shift v_1 and v_2 to the next available grid point along $\alpha_{(0,1)}$ and $\alpha_{(-1,-1)}$, respectively. For S_2 construct a three-semi-axes drawing Γ_{S_2} with axes $\alpha_{(-1,1)}$, $\alpha_{(2,-1)}$, and $\alpha_{(0,-1)}$, such that v_4 , v_5 , and v_6 lie on axes $\alpha_{(-1,1)}$, $\alpha_{(2,-1)}$, and $\alpha_{(0,-1)}$, respectively. Also, shift vertex v_5 to the next available grid point along $\alpha_{(2,-1)}$. Drawings Γ_{S_1} and Γ_{S_2} are combined in such a way that v_3 is one unit to the left of v_6 (see Fig. 3).

We now show that Γ is a planar drawing of G with area $\frac{2}{9}n^2 + O(n)$. First, observe that shifting vertices v_1 , v_2 , and v_5 does not compromise the planarity of Γ_{S_1} and Γ_{S_2} . Second, observe that, independently of the degrees of $v_1, \dots, v_6$ in G' , the edges between vertices of $t_{n/6}$ and vertices of $t_{n/6+1}$ do not intersect (see Figs. 2(c) and 2(d)). Concerning the area of Γ , we have that the height of Γ is equal to the height of Γ_{S_1} , which is $2\frac{n}{6} + 2$, while the width of Γ can be obtained by summing the horizontal lengths of edges (v_2, v_3) , (v_3, v_6) , and (v_6, v_5) , yielding $(2\frac{n}{6} + 1) + 1 + (2\frac{n}{6} + 2)$. Hence, the area of Γ is $(\frac{n}{3} + 2) \times (2\frac{n}{6} + 4) = \frac{2}{9}n^2 + O(n)$.

If n is odd, we add a triangle $t_0 > t_1$ in Γ^* , and arbitrarily augment Γ^* to a maximal nested triangles graph by adding dummy edges. Applying the construction described above we obtain a drawing with area $(\frac{n+3}{3} + 2) \times (2\frac{n+3}{3} + 4) = \frac{2}{9}n^2 + O(n)$. $\square$

Next, we prove a lower bound on the area needed by any straight-line drawing of a nested triangles graph when the outer face is not fixed.

Lemma 2. *Every n -vertex nested triangles graph requires $\frac{n^2}{9} + \Omega(n)$ area in any planar straight-line drawing Γ .*

Proof: Consider any nested triangles graph G admitting a planar drawing Γ^* in which $n/3$ disjoint triangles $t_1, t_2, \dots, t_{n/3}$ can be found such that $t_1 > t_2 > \dots > t_{n/3}$. Suppose that n is a multiple of 6. Choose any face f to be the outer face of Γ . Three are the cases: (i) $f = t_1$; (ii) $f = t_{n/3}$; and (iii) f is contained into the region of the plane delimited by t_k and t_{k+1} in Γ^* , for some $k \in \{1, \dots, n/3 - 1\}$. In Cases (i) and (ii), we have $n/3$ disjoint nested triangles in any planar drawing of G and, by Property 1, $(\frac{2n}{3} - 1)^2 = \frac{4}{9}n^2 + \Omega(n)$ area is required. In Case (iii), we have that in any planar drawing Γ of G there are two sequences S_1 and S_2 of nested triangles such that $S_1 : t_{k+1} > t_{k+2} > \dots > t_{n/3}$ and $S_2 : t_k > t_{k-1} > \dots > t_1$. One between S_1 and S_2 has at least $n/6$ nested triangles and, by Property 1, $(2\frac{n}{6} - 1)^2 = \frac{n^2}{9} + \Omega(n)$ area is required. $\square$

4 Conclusions and Open Problems

In this note we have shown that $2n^2/9 + O(n)$ area is sufficient and that $n^2/9 + \Omega(n)$ area is necessary to construct straight-line planar drawings of nested triangles graphs. Closing the gap between the upper and the lower bound is a natural open question. We conjecture the following:

Conjecture 1. Any maximal nested triangles graph requires $2n^2/9 + \Omega(n)$ area in any straight-line planar grid drawing.

Our conjecture is motivated by the following considerations: Choosing an arbitrary outer face for a maximal nested triangles graph composed of $n/3$ nested triangles produces two sequences S_1 and S_2 of h and $n/3 - h$ nested triangles, respectively. Such sequences lie in disjoint regions of the plane. Hence, it is possible to find a lower bound for the convex hull area of the whole drawing by summing up the convex hull areas of S_1 and S_2 . It appears to be the case that the area of the convex hull of h nested triangles is at least $2h^2 + \Omega(h)$ (the statement is

Fig. 4. (a) A plane graph and a minimum-area straight-line drawing of it. (b) Construction of a plane graph requiring $4n^2/9 - 2n/3$ area.

trivial when one side of the external triangle is parallel to one of the axes). Provided that the above statement is true, we can show that the minimum convex hull area of the whole drawing is $n^2/9 + \Omega(n)$, obtained for $h = n/6$. This would imply that the minimum area of the bounding box is $2n^2/9 + \Omega(n)$.

Renown problems in this field are those of determining the area required by straight-line drawings of general planar and plane graphs. Concerning the latter, in [6,9,1] it is reported the long-standing conjecture that the nested triangles graph is a worst case, i.e., that any plane graph can be drawn in $\lceil 2n/3 - 1 \rceil \times \lceil 2n/3 - 1 \rceil$ area, which, for $n \equiv 0 \pmod 3$, gives $4n^2/9 - 4n/3 + 1$. We remark that such a bound neglects at least a linear area term. Consider, in fact, the six-vertex graph G shown in Fig. 4(a). By case study we can prove that the smaller drawings of G have 2×6 and 3×4 bounding-boxes. The graph obtained by nesting G into $n/3 - 2$ nested triangles (see Fig. 4(b)) has at least $(2(n/3 - 2) + 3) \times (2(n/3 - 2) + 4) = (2n/3 - 1) \times (2n/3) = 4n^2/9 - 2n/3$ area.

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