Colorability in Orthogonal Graph Drawing^{*}

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Abstract. This paper studies the question: What is the maximum integer $k_{b,n}$ such that every $k_{b,n}$ -colorable graph has a b -bend n -dimensional orthogonal box drawing?

We give an exact answer for the orthogonal line drawing in all dimensions and for the 3-dimensional rectangle visibility representation. We present an upper and lower bound for the 3-dimensional orthogonal drawing by rectangles and general boxes. Particularly, we improve the best known upper bound for the 3-dimensional orthogonal box drawing from 183 to 42 and the lower bound from 3 to 22.

1 Introduction

The visualization of relational information has many applications in various domains. The domain entities are usually modeled as vertices and the relationships among entities are represented by edges.

There have been many graph drawing styles studied in the literature. In this paper we study the orthogonal box drawing. This drawing has received a wide attention recently due to its applications: 2-dimensional variants in VLSI routing, circuit board layout, CASE tools etc. and 3-dimensional variants for example in packaging algorithms [1,5,6,12].

The orthogonal box drawing represents vertices by axis-parallel boxes. Every edge is drawn as an axis-parallel polyline with ends on boundaries of boxes that correspond to vertices of the edge. Edges don't intersect other boxes and with the exception of the 2D drawing an edge cannot intersect another edge.

We call the drawing b -bend if each edge consists of at most $b+1$ line segments. A 0-bend drawing is called a straight-line drawing. If all edges in a straight-line box drawing in $\mathbb{R}^3$ are parallel then we can ignore the thickness of the boxes in this direction. We obtain a representation known as a 3D rectangle visibility drawing. The same operation in $\mathbb{R}^2$ gives us a bar-visibility drawing.

It turns out that the recognition of graphs with the given type of orthogonal drawing is difficult. For example, Shermer [7] shows that the recognition of graphs with 2-dimensional straight-line orthogonal drawing is NP-complete. Fekete et al. [8] establish NP-completeness of recognition of graphs with a 3D rectangle visibility drawing by squares.

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If we cannot effectively decide whether a graph has a drawing of the given type then it is natural to look for classes of graphs for which this decision is possible. The previous research was concentrated mainly on complete graphs e.g. on the determination of the maximum size of a complete graph with a drawing [2,3,4]. Unfortunately such results don't tell us much about drawing of graphs with more vertices.

Our search for a more practical class of graphs has been inspired by the open problem presented by Wood [1]:

What is the maximum $k \in \mathbb{Z}^+$ such that every k -colorable graph has a straight-line 3D orthogonal box drawing?

We study graphs with bounded colorability in this paper. Every k -colorable graph is a subgraph of a k -partite graph that is itself k -colorable. Therefore it is sufficient to study drawing of k -partite graphs.

Definition 1. *The multipartite number of the given type of drawing is the maximum $k \in \mathbb{N}$ such that every k -partite graph has a drawing of that type. We say that the multipartite number is infinite when every multipartite graph has such a drawing.*

Wood [1] proves that the multipartite number of the straight-line orthogonal box drawing is at least 3. On the other hand Fekete and Meijer [2] shows that it is at most 183.

We improve the lower bound from 3 to 22 and the upper bound from 183 to 42. We also determine the exact value of the multipartite number of the orthogonal drawing by line segments and of the rectangle visibility drawing. Table 1 summarizes the results presented in this work.

Table 1. Multipartite number of d -dimensional b -bend orthogonal drawing by v -dimensional boxes

v	d	b	multipartite number	
1	1	0	1	Theorem 1
	2	≥ 2	∞	Section 3.3
		1	2	Theorem 2
		0	1	Theorem 1
	3	≥ 1	∞	Theorem 2
	≥ 3	0	3	Theorem 1
2	2	≥ 1	∞	Section 4.3
		0	1	Section 4.2
	3	0	$\in \langle 22, 42 \rangle$ Theorems 4, 5	
3	3	≥ 1	∞	Section 5.2
		0	$\in \langle 22, 42 \rangle$ Theorems 4, 5	
rectangle visibility drawing			8	Theorem 3

2 Preliminaries

The next lemma is a simple application of the pigeon-hole principle. We include it because we use this formulation several times in the sequel.

Lemma 1. *Let $k, n, c \in \mathbb{N}$ and G be a complete k -partite graph whose each part has at least $c(n-1) + 1$ vertices and each vertex has one of c colors. Then G contains a complete k -partite subgraph whose each part is monochromatic and contains at least n vertices.*

Proof. Each part contains at least $c(n-1) + 1$ vertices. Therefore there are at least n vertices with the same color in each part. These monochromatic sets form the k -partite subgraph with the required properties. $\square$

We use this lemma when each vertex from a drawing must have one property from a finite set of properties and we have to ensure that the vertices from the same part have the same property. A similar situation occurs when the properties are assigned to edges.

Lemma 2. *Let $k, n, c \in \mathbb{N}$ and G be a complete k -partite graph whose each edge has one of c colors. There exists $N_{k,n,c} \in \mathbb{N}$ such that if each part of G has at least $N_{k,n,c}$ vertices then G contains a complete k -partite subgraph whose each part has at least n vertices and for each pair of parts the edges among elements of these parts are monochromatic.*

Proof. Recall that the bipartite Ramsey number $b_c(H)$ is the minimum m such that every c -coloring of $E(K_{m,m})$ yields a monochromatic copy of H . Chvátal [10] and Bieleke-Schwenk [11] proved that $b_c(K_{p,q}) \leq (q-1)c^p + O(c^{p-1})$.

If we fix two parts P_1 and P_2 that have at least $b_c(K_{n,n})$ vertices then there is a subgraph of G that is also complete k -partite, has at least n vertices from P_i , $i = 1, 2$ and the edges among these vertices are monochromatic. The required subgraph can be obtained by a repeatable application of this fact. $\square$

Sometimes we need to separate the parts with respect to some function on the set of vertices.

Lemma 3. *Let $k, n \in \mathbb{N}$ and $G(V, E)$ be a complete k -partite graph whose each part has at least $(n-1)k + 1$ vertices. For each $\ell : V \rightarrow \mathbb{R}$ there exists a complete k -partite subgraph G' of G whose each part has at least n vertices and whose parts are ℓ -separated e.g. for each pair P_1, P_2 of parts it is either $\forall x \in P_1 \forall y \in P_2 \ell(x) \leq \ell(y)$ or $\forall x \in P_1 \forall y \in P_2 \ell(y) \leq \ell(x)$.*

Proof. Sort the vertices $v \in V$ according to their value $\ell(v)$. Let P be the part whose n -th vertex (with respect to this order) has the lowest index in the sequence. Remove the vertices before the selected one from the sequence. Put the first n vertices of P into G' and remove the elements of P from the sequence. Continue in the same way until the sequence is empty.

Let's fix some part P . If P is not the selected part then at most $n-1$ of its vertices are removed from the sequence. Therefore at most $(n-1)(k-1)$ of

its vertices are removed before the part is selected. P has at least $(n - 1)k + 1$ elements. So, the described algorithm selects n vertices from each part. The resulting graph G' obviously has the required property. $\square$

Lemmas 1, 2 and 3 ensure the existence of a subgraph $G'(V', E')$ of a graph $G(V, E)$ such that $|V'| \geq f(|V|)$, where f is a non-decreasing function unbounded from above. These properties of f ensure that the size of G' can be made arbitrarily big if we take the original graph G sufficiently large. We use this fact many times in the sequel because we usually want to prove the existence of a large graph G' with some properties and are not interested in the exact size of G that must be taken to find a subgraph of the required size.

3 Line Drawing

3.1 Straight-Line Line Drawing

In this section we determine the multipartite number of the straight-line line drawing in the n -dimensional space e.g. the drawing where each vertex is represented by an axis-parallel line segment in $\mathbb{R}^n$.

Lemma 4. *The multipartite number of the straight-line line drawing in $\mathbb{R}^n$ is at most 3 for $n > 2$ and it is 1 for $n = 1, 2$.*

Proof. Let's suppose that we have a straight-line line drawing in $\mathbb{R}^n$ of a k -partite graph G . If we color the vertices according to their direction then Lemma 1 tells us that there exists a large k -partite subgraph G' with parallel vertices in the individual parts.

Now color the edges of the graph G' according to their direction. Let G'' denote the result of the application of Lemma 2 on the graph G' . The edges between arbitrary two parts of G'' are parallel.

We know that the vertices in the individual parts are parallel. We claim that the vertices from the different parts cannot be parallel. Suppose that the opposite holds e.g. there are parts P and Q such that the vertices from $P \cup Q$ are parallel (to a vector e_1). The edges between P and Q are parallel (to a vector e_2) due to the definition of G'' . This means that P and Q together with the edges between them lie in a plane (given by vectors e_1 and e_2). So, we have a bar-visibility graph of $K_{|P|, |Q|}$, but the sets P and Q can be made arbitrarily large. That is in a contradiction with the planarity of bar-visibility graphs.

If P and Q are two parts of G'' , P parallel to e_1 , Q parallel to e_2 and the edges between P and Q parallel to e_3 then the drawing of $K_{|P|, |Q|}$ lies in $S = p + e_1\mathbb{R} + e_2\mathbb{R} + e_3\mathbb{R}$, where p is a point of some line in $P \cup Q$. If $k > 3$ then there must exist a part R parallel to vector $e_4 \notin e_1\mathbb{R} + e_2\mathbb{R} + e_3\mathbb{R}$. The edges between P and R are parallel to a vector v .

Choose $l \in R$. $|l \cap S| \leq 1$ because $e_4 \perp S$.

If $l \cap S = \{q\}$ then the edges between P and R have one end in q . There can be at most two such edges (from the directions v and $-v$). Therefore $|P| \leq 2$, but P can be arbitrarily large.

If $l \cap S = \emptyset$ then $l + \mathbf{v} \mathbb{R}$ intersects S in at most one point. This means that l can be connected to at most one vertex from P and we have a contradiction with the size of P again. Hence $k \leq 3$.

The case $n = 1$ is obvious. If $n = 2$ then $G(V, E)$ is a union of two planar bar-visibility graphs and as such has less than $12|V|$ edges. Therefore no sufficiently large bipartite graph has a 2D straight-line line drawing. $\square$

The previous proof utilizes the fact that we can choose from an arbitrary complete multipartite graph a subgraph that is itself a complete multipartite graph, has some specific property and can be made arbitrarily large if the original graph is sufficiently big. The additional property allows us to simplify the proof. We use this method in many of the following proofs.

The lower bound on the multipartite number that matches our upper bound e.g. the construction of a 3-dimensional straight-line line drawing of $K_{a,b,c}$ for arbitrary positive integers a, b, c is given by Wood [1].

Theorem 1. *The multipartite number of the straight-line line drawing in $\mathbb{R}^n$ is 3 for $n > 2$ and it is 1 for $n = 1, 2$.*

3.2 1-Bend Line Drawing

Theorem 2. *The multipartite number of the 1-bend line drawing in $\mathbb{R}^n$ is 2 for $n = 2$ and is infinite for $n > 2$.*

Proof. The proof is similar to the proof of Theorem 1. See [9] for details. $\square$

3.3 2-Bend Line Drawing

The Figure 1 shows that every complete graph has a 2-bend 2-dimensional line drawing. Therefore the multipartite number of this drawing is infinite.

Fig. 1. 2-bend 2-dimensional line drawing of K_k

4 Rectangle Drawing

In this section we study rectangle drawing e.g. box drawing where vertices are represented by 2D boxes. We work with rectangles in parallel planes as if they were in the same plane. Operations on such rectangles should be understood as operations on the projections (into one of the planes) and the projection of the result of the operation (for example the intersection of some rectangles) back into the individual planes of the rectangles.

4.1 Rectangle Visibility Drawing

Definition 2. Let B be a box in an orthogonal drawing. $x^+(B)$ denotes the maximum coordinate of a point in the box B . Similarly we define x^- , y^+ , y^- , z^+ and z^- .

Note that $z^+ = z^-$ in the rectangle visibility drawing. We denote this function simply by z there.

Lemma 5. The multipartite number of the rectangle visibility drawing is at most 8.

Proof. Suppose that we have a rectangle visibility drawing of a complete k -partite graph G . Apply Lemma 3 on this graph consecutively with functions z , x^+ , x^- , y^+ and y^- . We obtain a graph G' with parts separated with respect to these functions.

Take an arbitrary part P_i of G' and sort its elements according to their z -coordinates. Due to Erdős-Szekeres theorem we can choose from this sequence a subsequence P'_i of length at least $|P_i|^{1/16}$ that is monotone in x^+ , x^- , y^+ and y^- coordinates. Denote by G'' the complete k -partite graph with parts P'_i . From the construction of G'' it is obvious that its parts can be made arbitrarily large if we take G with sufficiently large parts.

We claim that we can suppose that the orthogonal projections (along the z -axis) of rectangles from G'' have a common intersection. Rectangles from the different parts must intersect to be able to see each other. So, it is sufficient to show that each part has a common intersection. That happens if and only if each two rectangles from this part intersect.

Let P_1 be a part without a common intersection. There must be two elements $r_1, r_2 \in P_1$ that don't intersect. Without loss of generality it is $x^+(r_1) < x^-(r_2)$.

Fig. 2.

G'' is a complete k -partite graph. Hence a rectangle r from a different part (to see both r_1 and r_2) must have $x^-(r) < x^+(r_1)$ and $x^+(r) > x^-(r_2)$.

Let's modify the part P_1 to have a common intersection. Let c_x^+ (respectively c_x^-) be a maximum (resp. minimum) x -coordinate of a point of a rectangle in P_1 . Denote by S the y -parallel strip between c_x^- and c_x^+ (see Figure 2).

The proved inequalities together with the x^+ , x^- -separability of parts ensures that $x^-(r) < c_x^-$ and $c_x^+ < x^+(r)$ for each rectangle $r \notin P_1$. Therefore if two

Fig. 3. Modification of stairs

rectangles not in P_1 can see each other through a point (x, y) in the strip S then they can see each other also through a point $((c_x^- - \varepsilon, y)$ or $(c_x^+ + \varepsilon, y)$ for a sufficiently small $\varepsilon > 0$) outside the strip.

The rectangles in P_1 (ordered according to the z -coordinate) are monotone in x^+ and x^- coordinates and don't have a common intersection. So, they must be either increasing or decreasing in both these coordinates - the rectangles form stairs (see Figure 3).

We claim that if we change x^+ and x^- coordinates of the stair rectangles to ensure a common intersection (as shown in Figure 3) then we don't destroy the completeness of the k -partite visibility representation of G'' .

Only the rectangles in the strip S are modified. So, the visibility among rectangles not in P_1 is not affected because they can see each other through points outside the strip. It remains to show that the rectangles from P_1 can see all other rectangles.

Sides y^+ and y^- of each rectangle not in P_1 cross the whole width of the strip S . They mark on the strip (orthogonal) sub-strips. The rectangles not in P_1 can see the rectangles from P_1 only through their sub-strips.

No visibility is destroyed if we move the x^+ and x^- coordinates of rectangles from P_1 such that the same rectangles remain visible through each sub-strip, but that is exactly what our stair-modification technique does.

We have shown that we can expect each part of G'' to have a common intersection.

We know that if we sort the rectangles from some part P of G'' according to their z -coordinates then we obtain a sequence monotone also in x^+, x^-, y^+ and y^- coordinates. Moreover if P has a common intersection then we can consider P to form a frame with sides oriented up and down (see Figure 4). The orientation determines the direction from which the corresponding sides of rectangles are visible.

We also know that the parts are x^+, x^-, y^+, y^- -separated. Thus two corresponding sides of frames cannot intersect (see Figure 5). The interiors of frames

Fig. 4. Transformation of one part into a frame and an oriented rectangle

intersect because all rectangles have a common intersection. Due to these facts we can shrink the frames into rectangles with oriented sides. Now two parts can see each other if the boundaries of their oriented rectangles intersect and the intersecting sides have a correct orientation.

Fig. 5. Examples of invalid (a), (b) and valid (c), (d) intersections of frames

It remains to prove that a complete graph with this modified oriented rectangle visibility representation has at most 8 vertices.

Lemma 6. *If K_n has a modified visibility representation by rectangles with oriented sides (as described in the previous proof) then $n \leq 8$.*

Proof. We proceed in a similar way as Fekete et al. [3] in the proof of the non-existence of a visibility representation of K_8 by unit squares e.g. by a computer search. Our algorithm is based on their algorithm. We modify it to generate visibility representations with general rectangles (not only squares). We also add a code that assigns an orientation to the individual sides. When the next rectangle is added the new code also checks whether the orientation requirements are satisfied. See [9] for details (including the source code).

We were able to process all valid configurations in 26 hours on Intel Centrino 1.7 GHz machine and verified that there is no representation of K_9 with the required properties.¹ $\square$

Lemma 7. *The multipartite number of the rectangle visibility drawing is at least 8.*

Proof. The Figure 6 shows (in the form of oriented rectangles) a rectangle visibility drawing of a complete 8-partite graph. The numbers in the lower right corner determine the order of parts with respect to the z -coordinate. The thick sides are oriented up, the thin sides are oriented down. The small circles show areas where the individual parts see each other. $\square$

If we put together Lemmas 5, 6 and 7 we obtain the following theorem.

Theorem 3. *The multipartite number of the rectangle visibility drawing is 8.*

¹ The algorithm was run several times on different hardware configurations in fact. Check sums were used to recognize computations affected by a potential hardware failure.

Fig. 6. Rectangle visibility representation of $K_{a,b,c,d,e,f,g,h}$

4.2 Straight-Line Rectangle Drawing

A graph with a 2-dimensional straight-line rectangle drawing is a union of two planar bar-visibility graphs. Therefore the multipartite number of such a drawing is one.

The 3-dimensional straight-line rectangle drawing has similar properties to the 3-dimensional straight-line box drawing. The upper bound on the multipartite number of the 3-dimensional straight-line box drawing proved in the next section is also the best known bound for the straight-line rectangle drawing. On the other hand the following drawing of a complete 22-partite graph provides also the best known lower bound on the multipartite number of the 3-dimensional straight-line box drawing.

Theorem 4. *The multipartite number of the straight-line rectangle drawing in $\mathbb{R}^3$ is at least 22.*

Proof. The Figure 7 shows a rectangle visibility drawing of a complete 6-partite graph. Take a copy of the construction from the Figure 6 and rotate it to be parallel to xz -plane. Place the copy between the third and the fourth part. Ensure that x^+ (resp. x^-) coordinates of the rectangles from the copy are bigger (resp. smaller) than the coordinates of the rectangles from the 6-partite graph.

Take another copy of the construction from the Figure 6 and rotate it to be parallel to yz -plane. Place the copy again between the third and the fourth part either below or over the first copy. Ensure that y^+ (resp. y^-) coordinates of the

Fig. 7. Straight-line rectangle drawing of a complete 22-partite graph

rectangles from this copy are bigger (resp. smaller) than the coordinates of the rectangles from the 6-partite graph.

The copies are schematically displayed on the Figure 7 by the horizontal and the vertical line. The small circles show the areas where the individual parts see each other. It can be easily verified that the resulting construction is a straight-line rectangle drawing of a complete 22-partite graph. $\square$

4.3 1-Bend Rectangle Drawing

It can be shown (using a construction similar to Figure 1) that the multipartite number of 1-bend rectangle drawing in $\mathbb{R}^2$ is infinite – if we place the rectangles on a diagonal then we need only one bend to connect any pair of rectangles.

5 Box Drawing

5.1 Straight-Line Box Drawing

Definition 3. Let A and B be axis-parallel boxes in $\mathbb{R}^3$. We write $A \prec_x B$ if A and B can see each other and $x^+(A) < x^-(B)$. Similarly we define $A \prec_y B$ and $A \prec_z B$.

It can be easily verified that these relations are partial orders on the boxes in a straight-line box drawing of a complete graph. We say that some boxes from a drawing form an x -chain (resp. x -antichain) if they form a chain (resp. antichain) in the partial order $\prec_x$.

Fekete and Meijer [2] showed the following properties of these orders.

Lemma 8. *Let C be a maximum length x -chain in a 3-dimensional orthogonal box drawing of a complete graph. There cannot be an x -chain D of length greater than 4 such that $C \cap D = \emptyset$.*

Lemma 9. *If there is no chain (x -chain, y -chain or z -chain) longer than 4 in a 3-dimensional orthogonal box drawing of a complete graph G then G can have at most 18 vertices.*

Lemmas 8 and 9 allow us to estimate the maximum size of a complete graph with a drawing with the bounded length of chains.

Lemma 10. *If k is the maximum length of a chain that appears in the partial order $\prec_x$, $\prec_y$ or $\prec_z$ in a 3-dimensional straight-line box drawing of K_n then $n \leq 3k + 18$.*

Proof. Let C_x , C_y resp. C_z denotes the maximum x -chain, y -chain resp. z -chain in the drawing. If we remove the boxes in $C_x \cup C_y \cup C_z$ from the drawing then by Lemma 8 there cannot remain a chain of length 5. There remain at most 18 boxes by Lemma 9. Hence, $n \leq |C_x| + |C_y| + |C_z| + 18 \leq 3k + 18$. $\square$

The presented properties of the partial orders $\prec_x$, $\prec_y$ and $\prec_z$ can be utilized to prove an upper bound for the multipartite number of a box drawing.

Theorem 5. *The multipartite number of the straight-line box drawing in $\mathbb{R}^3$ is at most 42.*

Proof. Let G be a large complete k -partite graph that has a 3D straight-line box drawing. Color its edges by three colors according to their direction. Apply Lemma 2 on G and denote by G' the resulting graph.

Select one box from each part of G' . These boxes form a 3D straight-line box drawing of K_k . We claim that this drawing doesn't contain a chain of length greater than 8. If we prove this then by the previous lemma $k \leq 3.8 + 18 = 42$.

Suppose by contradiction that there exists an x -chain of length 9. The boxes from the parts with a member in this chain also (due to the selection of G') see each other along the x -axis. Therefore they correspond to a rectangle visibility representation of a 9-partite graph that can be made arbitrarily large if G is taken sufficiently large. This is in a contradiction with Theorem 3. $\square$

5.2 1-Bend Box Drawing

The infinity of the multipartite number of the 1-bend box drawing comes immediately from the infinity for the 1-bend 3D line drawing.

6 Conclusion

We determine the multipartite number of several types of drawings. This significantly enlarges the class of graphs that are known to have such drawings. For example Fekete and Meijer show in [2] that each graph with at most 56 vertices has a 3-dimensional straight-line orthogonal box drawing. Comparing to this we prove that any 22-colorable graph has such a drawing.

References

1. Wood, D.R.: Three-Dimensional Orthogonal Graph Drawing, Ph.D. Thesis, School of Computer Science and Software Engineering, Monash University (2000)
2. Fekete, S.P., Meijer, H.: Rectangle and box visibility graphs in 3D. *Int. J. Comput. Geom. Appl.* 97, 1–28 (1997)
3. Fekete, S.P., Houle, M.E., Whitesides, S.: New results on a visibility representation of graphs in 3D. In: *Proc. Graph Drawing*, vol. 95, pp. 234–241 (1995)
4. Bose, P., Dean, A., Hutchinson, J., Shermer, T.: On rectangle visibility graphs. In: *Proc. Graph Drawing*, vol. 96, pp. 25–44 (1996)
5. Hutchinson, J.P., Shermer, T., Vince, A.: On representations of some thickness-two graphs. *Comput. Geom.* 13(3), 161–171 (1999)
6. Dean, A.M., Hutchinson, J.P.: Rectangle-visibility Layouts of Unions and Products of Trees. *J. Graph Algorithms Appl.*, 1–21 (1998)
7. Shermer, T.: Block visibility representations III: External visibility and complexity. In: Fiala, K., Sack (eds.) *Proc. of 8th Canadian Conf. on Comput. Geom. Int. Informatics Series*, vol. 5, pp. 234–239. Carleton University Press, Ottawa (1996)
8. Fekete, S.P., Houle, M.E., Whitesides, S.: The wobbly logic engine: proving hardness of non-rigid geometric graph representation problems, Report No. 97.273, Angewandte Mathematik und Informatik Universität zu Köln (1997)
9. Štola, J.: Chromatic invariants in graph drawing, Master's Thesis, Department of Applied Mathematics, Charles University, Prague, Czech Republic (2006), <http://kam.mff.cuni.cz/~stola/chromaticInvariants.pdf>
10. Chvátal, V.: On finite polarized partition relations. *Canad. Math. Bul.* 12, 321–326 (1969)
11. Beineke, L.W., Schwenk, A.J.: On a bipartite form of the Ramsey problem. In: *Proc. 5th British Combin. Conf. 1975, Congr. Numer.*, vol. XV, pp. 17–22 (1975)
12. Biedl, T., Shermer, T., Whitesides, S., Wismath, S.: Bounds for orthogonal 3D graph drawing. *J. Graph Alg. Appl.* 3(4), 63–79 (1999)