Straight-Line Orthogonal Drawings of Binary and Ternary Trees^{*}

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Abstract. In this paper we provide upper and lower bounds on the area requirement of straight-line orthogonal drawings of n -node binary and ternary trees. Namely, we show algorithms for constructing order-preserving straight-line orthogonal drawings of binary trees in $O(n^{1.5})$ area, straight-line orthogonal drawings of ternary trees in $O(n^{1.631})$ area, and straight-line orthogonal drawings of complete ternary trees in $O(n^{1.262})$ area. As far as we know, the ones we present are the first algorithms achieving sub-quadratic area for these problems. Further, for upward order-preserving straight-line orthogonal drawings of binary trees and for order-preserving straight-line orthogonal drawings of ternary trees we provide $\Omega(n^2)$ area lower bounds, that we also prove to be tight.

1 Introduction

The design of algorithms for constructing orthogonal and straight-line drawings of binary and ternary trees, that are trees whose maximum degree is bounded by three and four, respectively, has attracted considerable research efforts in the Graph Drawing community. Orthogonal and straight-line planar drawings are easily readable by the viewer and hence they are among the most studied drawing standards. When dealing with orthogonal or straight-line tree drawings, it is common to consider area minimization as an important aesthetic requirement to satisfy. The study of area minimization for binary and ternary tree drawings has been motivated by VLSI circuits design and it is still attractive for the sake of rendering acyclic relationships on a screen limited by a finite resolution rule. Nevertheless, the beauty of some combinatorial and geometric open problems concerning area minimization of straight-line and orthogonal drawings of trees justifies their study even looking at them from a purely theoretical point of view.

Almost thirty years ago, Valiant proved in [12] that every n -node ternary tree admits a $\Theta(n)$ area orthogonal drawing. Such a result was strengthened in [5], where Dolev and Trickey proved that ternary trees admit $\Theta(n)$ area order-preserving orthogonal drawings. A $\Theta(n \log \log n)$ optimal bound for upward orthogonal drawings of binary trees was proved by Garg et al. in [6], while in [9] Kim showed that $\Theta(n \log n)$ area is an optimal bound for upward orthogonal

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drawings of ternary trees. Concerning the area requirement of planar straight-line drawings, Garg and Rusu proved in [8] that linear area suffices for bounded degree trees, while $\Theta(n \log n)$ area is asymptotically optimal if the drawing is required to be upward and order-preserving [7].

Drawings that are simultaneously straight-line and orthogonal provide extremely high readability of the combinatorial structure of a tree, and hence it is a serious lack in the literature that only few results concerning area minimization of straight-line orthogonal drawings of binary and ternary trees are known. Chan et al. in [1], and Shin et al. in [11] have shown that $O(n \log \log n)$ area suffices for straight-line orthogonal drawings of binary trees. Further, it has been shown in [3,1] that binary trees admit upward straight-line orthogonal drawings in $O(n \log n)$ area. Such an area bound is worst-case optimal, as proved in [1].

In this paper we present the following results: (i) order-preserving straight-line orthogonal drawings of binary trees can be constructed in $O(n^{1.5})$ area (Section 3); (ii) upward order-preserving straight-line orthogonal drawings of binary trees require (and can be realized in) $\Omega(n^2)$ area (Section 3); (iii) straight-line orthogonal drawings of ternary trees can be constructed in $O(n^{1.631})$ area (Section 4); (iv) order-preserving straight-line orthogonal drawings of ternary trees require (and can be realized in) $\Omega(n^2)$ area (Section 4); (v) straight-line orthogonal drawings of complete ternary trees can be constructed in $O(n^{1.262})$ area (Section 5); and (vi) there exist ternary trees for which the minimum side of any straight-line orthogonal drawing is $\Omega(n^{0.438})$ and, for complete ternary trees, such a bound is tight (Section 5).

Table 1. Summary of the best known area bounds for straight-line orthogonal drawings of binary and ternary trees. For complete trees the order-preserving column is not considered, since such trees are symmetric. Straight-line orthogonal upward drawings of ternary trees cannot generally be constructed.

	Upward	Order-preserving	Upper Bound	Ref.	Lower Bound	Ref.
Complete Binary	✓		$O(n)$	[3]	$\Omega(n)$	trivial
Complete Binary			$O(n)$	[10]	$\Omega(n)$	trivial
Binary	✓	✓	$O(n^2)$	[3]	$\Omega(n^2)$	Th. 1
Binary	✓		$O(n \log n)$	[3,1]	$\Omega(n \log n)$	[1]
Binary		✓	$O(n^{1.5})$	Th. 2	$\Omega(n)$	trivial
Binary			$O(n \log \log n)$	[1,11]	$\Omega(n)$	trivial
Complete Ternary	✓		non-drawable			
Complete Ternary			$O(n^{1.262})$	Th. 6	$\Omega(n)$	trivial
Ternary	✓	✓	non-drawable			
Ternary	✓		non-drawable			
Ternary		✓	$O(n^2)$	Th. 4	$\Omega(n^2)$	Th. 3
Ternary			$O(n^{1.631})$	Th. 5	$\Omega(n)$	trivial

2 Preliminaries

We assume familiarity with trees and their drawings (see also [4]).

A *rooted tree* T is a tree with one distinguished node, called *root* and denoted by $r(T)$. In the following we assume that binary and ternary trees are rooted

at any node of degree at most two and three, respectively. A *spine* in T is a path connecting $r(T)$ to a leaf. A *double-spine* in T is a path connecting two leaves and passing through $r(T)$. A tree is *ordered* if an order of the children of each node is specified. For an ordered binary tree we talk about *left* and *right child*. For an ordered ternary tree we talk about *left*, *middle*, and *right child*. The subtrees rooted at the left, middle, and right child of a node u are the *left*, *middle*, and *right subtree* of u , respectively. The subtree of a given tree rooted at node u is denoted by $T(u)$. Removing a path $\mathcal{P}$ from a tree disconnects the tree into connected components. The ones containing children of nodes in $\mathcal{P}$ are subtrees of $\mathcal{P}$. If the tree is ordered, then each component is a left, middle, or right subtree of $\mathcal{P}$, depending on whether the root of such subtree is a left, middle, or right child of a node in $\mathcal{P}$. We denote by $|T|$ the number of nodes in a tree T . The *heaviest* tree in a set of trees is the one with the greatest number of nodes. A *complete tree* is such that all non-leaf nodes have the same degree and all spines have the same number of nodes, called the *height* of the tree.

A *straight-line orthogonal grid drawing* of a binary or ternary tree is a mapping of its nodes to distinct points with integer coordinates and of its edges to horizontal or vertical segments between such points. A drawing is *planar* if no two segments cross, but, possibly, at common end-points. In the following we use *SO-drawing* as short for straight-line orthogonal planar grid drawing. An SO-drawing is *upward* if every node is drawn not below its children. An SO-drawing Γ is *order-preserving* if, for every node u , the segments connecting u to its left child, middle child, right child and parent appear in Γ in this order around u . When we talk about order-preserving drawings, we suppose that trees are ordered. Consider an SO-drawing Γ of a rooted tree T . Denote by l the vertical half-line starting at $r(T)$ and directed upward. Then Γ has the *top visibility property* if no node, but for $r(T)$, is placed on l and no edge crosses l . Denote by r the horizontal line through $r(T)$. Then Γ has the *side visibility property* if no node, but for $r(T)$, is placed on r and no edge crosses r . The *width* (*height*) of a drawing is the number of vertical (horizontal) grid lines intersecting it. The *area* of a drawing is its height multiplied by its width.

3 Straight-Line Orthogonal Order-Preserving Drawings of Binary Trees

First, we show that order-preserving upward SO-drawings of binary trees generally require quadratic area. Such a bound is matched by an $O(n^2)$ upper bound obtained by using the well-known *h-v layout* (see, e.g., [3]).

Theorem 1. *There exists an n -node binary tree T requiring $\Omega(n^2)$ area in any upward order-preserving SO-drawing.*

Proof: Assume $n \equiv 0 \pmod 6$. Tree T is composed of (see Fig. 1.a): (i) an $n/6$ -node spine $C_1 : (m_0 = r(T), m_1, \dots, m_{\frac{n}{6}-2}, m_{\frac{n}{6}-1})$, with m_i left child of m_{i-1} , for $1 \leq i \leq \frac{n}{6} - 1$; (ii) an $n/6$ -node spine $C_2 : (p_0 = r(T), p_1, \dots, p_{\frac{n}{6}-2}, p_{\frac{n}{6}-1})$, with p_i right child of p_{i-1} , for $1 \leq i \leq \frac{n}{6} - 1$; (iii) the right child m_i^r of each

Fig. 1. (a) Tree T providing the lower bound of Theorem 1. (b)-(c)-(d) Possible placements of $r(T)$ and its children.

node m_i of C_1 , with $1 \leq i \leq \frac{n}{6} - 2$; (iv) the left child p_i^l of each node p_i of C_2 , with $1 \leq i \leq \frac{n}{6} - 2$; (v) a path C_3 of $n/6 + 3$ nodes, alternating between right and left children, such that one end-vertex of C_3 is m_1^r ; and (vi) $n/6 + 3$ leaves attached to C_3 , alternating between left and right children.

Consider any upward order-preserving SO-drawing Γ of T . In [6] it is shown that C_3 and its attached leaves require $\Omega(n)$ height in any upward order-preserving drawing. Consider the relative position of $r(T)$ and its children in Γ . Three are the cases; either m_1 is to the left of $r(T)$ and p_1 is below $r(T)$ (see Fig. 1.b), or m_1 is below $r(T)$ and p_1 is to the right of $r(T)$ (see Fig. 1.c), or m_1 is to the left of $r(T)$ and p_1 is to the right of $r(T)$ (see Fig. 1.d). Suppose m_1 is to the left of $r(T)$. We prove by induction that each node m_i of C_1 , with $1 \leq i \leq \frac{n}{6} - 1$, is drawn at least one unit to the left of its parent. The claim holds in the base case by the assumption that m_1 is to the left of $m_0 = r(T)$. If m_i is to the left of m_{i-1} , then the edges from m_i to its children are drawn towards the left and the bottom. Since the drawing is order-preserving, m_i^r must be below m_i and m_{i+1} to the left of m_i . So each node m_i , with $1 \leq i \leq \frac{n}{6} - 1$, is drawn at least one unit to the left of its parent, implying a linear lower bound on the width of Γ . If m_1 is not to the left of $r(T)$ then p_1 is to the right of $r(T)$ and a similar argument shows that each node p_i , with $1 \leq i \leq \frac{n}{6} - 1$, is at least one unit to the right of its parent, again implying a linear lower bound on the width of Γ . Hence both the height and the width of Γ are $\Omega(n)$. $\square$

Now we turn to non-upward drawings, showing that sub-quadratic area suffices for order-preserving SO-drawings:

Theorem 2. Any n -node binary tree T admits an $O(n^{1.5})$ area order-preserving SO-drawing.

Proof: We describe an inductive algorithm constructing an order-preserving SO-drawing Γ of T satisfying the side visibility property. If $n = 1$, then Γ is trivially constructed. Suppose $n > 1$. Select a double-spine $\pi = (u_k, u_{k-1}, \dots, u_1, u_0 = r(T) = v_0, v_1, \dots, v_m)$ in T . How to choose π is discussed later. Denote by p_i the non-spine child of a node $u_i \in \pi$ and by q_j the non-spine child of a node $v_j \in \pi$.

Fig. 2. Illustrations for the algorithm in the proof of Theorem 2. Left (right) edges are labeled $l(r)$. Label $l(r)$ inside a subtree shows the direction of the edge from the root to its left (right) child.

Recursively construct drawings $\Gamma(p_i)$ of $T(p_i)$ and $\Gamma(q_j)$ of $T(q_j)$ satisfying the side visibility property, for $1 \leq i < k$ and $1 \leq j < m$. Let h_v , h_v^{-1} , and h_v^1 be vertical grid lines with h_v^{-1} (h_v^1) one unit to the left (to the right) of h_v . Draw $r(T)$ on h_v . For $i = 1, 2, \dots, k-1$, if p_i is the left child of u_i rotate $\Gamma(p_i)$ of π and place it so that the rightmost vertical line intersecting it is h_v^{-1} and with the lowest horizontal line intersecting it one unit above the highest horizontal line intersecting $\Gamma(p_{i-1})$ or u_{i-1} ; otherwise (p_i is the right child of u_i), place $\Gamma(p_i)$ so that the leftmost vertical line intersecting it is h_v^1 and with the lowest horizontal line intersecting it one unit above the highest horizontal line intersecting $\Gamma(p_{i-1})$ or u_{i-1} . Draw u_i on h_v on the same horizontal line of its already drawn child (or one unit above the highest horizontal line intersecting $\Gamma(p_{i-1})$ or u_{i-1} if no child of u_i has been drawn). Draw u_k on h_v one unit above the highest horizontal line intersecting $\Gamma(p_{k-1})$ or u_{k-1} . For $j = 1, 2, \dots, m-1$, if q_j is the right child of v_j rotate $\Gamma(q_j)$ of π and place it so that the rightmost vertical line intersecting it is h_v^{-1} and with the highest horizontal line intersecting it one unit below the lowest horizontal line intersecting $\Gamma(q_{j-1})$ or v_{j-1} ; otherwise (q_j is the left child of v_j), place $\Gamma(q_j)$ so that the leftmost vertical line intersecting it is h_v^1 and with the highest horizontal line intersecting it one unit below the lowest horizontal line intersecting $\Gamma(q_{j-1})$ or v_{j-1} . Draw v_j on h_v on the same horizontal line of its already drawn child (or one unit below the lowest horizontal line intersecting $\Gamma(q_{j-1})$ or v_{j-1} if no child of v_j has been drawn). Draw v_m on h_v one unit below the lowest horizontal line intersecting $\Gamma(q_{m-1})$ or v_{m-1} (see Fig. 2.a).

It's easy to see that the constructed drawing Γ is an order-preserving SO-drawing satisfying the side visibility property. Let's analyze the area requirement of Γ . Concerning its height, there is at least one node of T on each horizontal grid line intersecting Γ , hence the height of Γ is $O(n)$. Denote by $W(T)$ the width of the drawing constructed by the described algorithm when its input is binary tree T . Let also $W(n) = \max\{W(T)\}$ over all binary trees T with n

nodes. Since all subtrees drawn to the left (to the right) of π are aligned on their right side (on their left side) and since $W(n)$ is a non-decreasing function of n , then $W(n) = W(n_l) + W(n_r) + 1$, where n_l (n_r) is the number of nodes in the heaviest subtree drawn to the left (to the right) of π . To get a good bound for $W(n)$ we need to carefully choose π . A technique similar to the one we present was introduced in [2] for selecting (single) spines. π is composed of two spines $\mathcal{U} = (u_0, u_1, \dots, u_k)$ and $\mathcal{V} = (v_0, v_1, \dots, v_m)$. Spine $\mathcal{U}$ is iteratively selected as follows: $u_0 = r(T)$, u_1 is the left child of u_0 . Denote by l_i and by r_i the left and right child of u_i , respectively. Denote also by α_i and by β_i the heaviest left subtree and the heaviest right subtree of path $(u_1, \dots, u_{i-1})$ (see Fig. 2.b). If $|\alpha_i| + |T(r_i)| \leq |\beta_i| + |T(l_i)|$ then set $u_{i+1} = l_i$, otherwise set $u_{i+1} = r_i$. Spine $\mathcal{V}$ is iteratively selected as follows: $v_0 = r(T)$, v_1 is the right child of u_0 . Denote by l_j and by r_j the left and right child of v_j , respectively. Denote by α_j the one between the heaviest right subtree of path $(v_1, \dots, v_{j-1})$ and the heaviest left subtree of $\mathcal{U} \setminus u_0$ that has the greatest number of nodes. Denote also by β_j the one between the heaviest left subtree of path $(v_1, \dots, v_{j-1})$ and the heaviest right subtree of $\mathcal{U} \setminus u_0$ that has the greatest number of nodes (see Fig. 2.c). If $|\alpha_j| + |T(l_j)| \leq |\beta_j| + |T(r_j)|$ then set $v_{j+1} = r_j$, otherwise set $v_{j+1} = l_j$. Similarly to [2], we get the following:

Lemma 1. *For any left subtree α of $\mathcal{U} \setminus u_0$ or right subtree α of $\mathcal{V} \setminus v_0$ and for any right subtree β of $\mathcal{U} \setminus u_0$ or left subtree β of $\mathcal{V} \setminus v_0$, $|\alpha| + |\beta| \leq n/2$.*

Proof: If α and β are both subtrees of $\mathcal{U} \setminus u_0$ or if are both subtrees of $\mathcal{V} \setminus v_0$, then the statement follows as in Lemma 4.1 of [2]. Otherwise, suppose α is a left subtree of $\mathcal{U} \setminus u_0$ and β is a left subtree of $\mathcal{V} \setminus v_0$. Let v_j be the parent of β 's root. Denote by l_j and r_j the left and right child of v_j , respectively. Notice that $r_j = v_{j+1}$. Denote by α_j the one between the heaviest right subtree of path $(v_1, \dots, v_{j-1})$ and the heaviest left subtree of $\mathcal{U} \setminus u_0$ that has the greatest number of nodes, and denote by β_j the one between the heaviest left subtree of path $(v_1, \dots, v_{j-1})$ and the heaviest right subtree of $\mathcal{U} \setminus u_0$ that has the greatest number of nodes. By construction $|\alpha_j| + |T(l_j)| \leq |\beta_j| + |T(r_j)|$. Moreover, $|\alpha_j| + |T(l_j)| + |\beta_j| + |T(r_j)| \leq n$. Therefore, $\alpha_j + |T(l_j)| \leq n/2$. Since $\alpha \leq \alpha_j$ and $\beta = T(l_j)$, the statement follows. The case in which α is a right subtree of $\mathcal{V} \setminus v_0$ and β is a right subtree of $\mathcal{U} \setminus u_0$ is analogous. $\square$

Selecting π as just described, we get $W(n) \leq \max_{n_1+n_2 \leq n/2} W(n_1) + W(n_2) + 1$. As already noticed in [2], by Hölder's inequality $n_1 + n_2 \leq n/2$ implies $\sqrt{n_1} + \sqrt{n_2} \leq \sqrt{n}$ and, by induction, $W(n) \leq c\sqrt{n} - 1$, for some constant c depending only on the values of $W(n)$ with n small. $\square$

4 Straight-Line Orthogonal Drawings of Ternary Trees

In this section we consider SO-drawings of ternary trees. First, we show that if an order of the children of each node is fixed, then quadratic area is necessary in the worst case.

Theorem 3. *There exists an n -node ternary tree T requiring $\Omega(n^2)$ area in any order-preserving SO-drawing.*

Proof: Assume $n \equiv 4 \pmod 9$. Tree T is composed of (see Fig. 3.a): (i) a spine $C_1 : (m_0 = r(T), m_1, \dots, m_{\frac{n-4}{9}}, m_{\frac{n+5}{9}})$, with m_1 left child of $r(T)$ and m_i middle child of m_{i-1} , for $i = 2, 3, \dots, \frac{n+5}{9}$; (ii) a spine $C_2 : (p_0 = r(T), p_1, \dots, p_{\frac{n-4}{9}}, p_{\frac{n+5}{9}})$, with p_i middle child of p_{i-1} , for $i = 1, 2, \dots, \frac{n+5}{9}$; (iii) a spine $C_3 : (q_0 = r(T), q_1, \dots, q_{\frac{n-4}{9}}, q_{\frac{n+5}{9}})$, with q_1 right child of $r(T)$ and q_i middle child of q_{i-1} , for $i = 2, 3, \dots, \frac{n+5}{9}$; and (iv) a left and a right child for each node m_i , p_i , and q_i , for $i = 1, 2, \dots, \frac{n-4}{9}$. Consider any order-preserving SO-drawing of C_1 and of the children of nodes m_i , with $1 \leq i \leq \frac{n-4}{9}$. Suppose that m_1 is to the left of m_0 . Then, to preserve the order of the children of m_1 , m_i is to the left of m_{i-1} , for $i = 2, 3, \dots, \frac{n+5}{9}$. Analogously, if m_1 is to the right, above, or below m_0 , then m_i is to the right, above, or below m_{i-1} , respectively, for $i = 2, 3, \dots, \frac{n+5}{9}$. Such an argument applies to C_2 (to C_3), as well: If p_1 (q_1) is to the left, to the right, above, or below p_0 (q_0), then p_i (q_i) is to the left, to the right, above, or below p_{i-1} (q_{i-1}), respectively, for $i = 2, 3, \dots, \frac{n+5}{9}$. Since $r(T)$ has three children, then at least one of them is above or below $r(T)$ and one of them is to the left or to the right of $r(T)$. Hence, any order-preserving SO-drawing of T has at least $\frac{n+5}{9} + 1$ height and width. $\square$

The proved bound is tight, as shown in the following:

Theorem 4. *Any n -node ternary tree T admits an $O(n^2)$ area order-preserving SO-drawing.*

Proof: We show an inductive algorithm constructing an order-preserving SO-drawing Γ of T satisfying the top visibility property. If $n = 1$, then Γ is trivially constructed. Suppose $n > 1$. Let T_l , T_m , and T_r be the left, middle, and right subtree of $r(T)$. By induction, drawings Γ_l , Γ_m , and Γ_r satisfying the top visibility property can be constructed for T_l , T_m , and T_r , respectively. Draw $r(T)$ in the plane. Rotate Γ_l of $\pi/2$ in clockwise direction. Place Γ_l with the rightmost vertical line intersecting it one unit to the left of $r(T)$ and with $r(T_l)$ on the same horizontal line of $r(T)$. Rotate Γ_r of $\pi/2$ in counter-clockwise direction. Place Γ_r with the leftmost vertical line intersecting it one unit to the right of $r(T)$ and with $r(T_r)$ on the same horizontal line of $r(T)$. Place Γ_m with the highest horizontal line intersecting it one unit below the lowest horizontal line intersecting Γ_l or Γ_r and with $r(T_m)$ on the same vertical line of $r(T)$ (see Fig. 3.b). It's easy to see that Γ is an order-preserving SO-drawing satisfying the top visibility property. Since Γ has at least one node for each horizontal and vertical grid line intersecting it, then its height and its width are $O(n)$. $\square$

For non-order-preserving drawings better bounds can be achieved:

Theorem 5. *Any n -node ternary tree T admits an $O(n^{1.631})$ area SO-drawing.*

Proof: We show an inductive algorithm that constructs an SO-drawing Γ of T satisfying the top visibility property. If $n = 1$, then Γ is trivially constructed.

Fig. 3. (a) Tree T requiring $\Omega(n^2)$ area in any order-preserving SO-drawing. (b) Illustration for the algorithm in Theorem 4. (c) Illustration for the algorithm in Theorem 5. Subtrees labeled by *second* or *third* are second or third heaviest subtrees, respectively.

Suppose $n > 1$. Select a double-spine $\pi = (u_k, u_{k-1}, \dots, u_1, u_0 = r(T) = v_0, v_1, \dots, v_m)$ in T such that: $T(v_1)$ is the heaviest subtree of $r(T)$; for $j = 2, 3, \dots, m$, $T(v_j)$ is the heaviest subtree of v_{j-1} ; $T(u_1)$ is the heaviest subtree of $r(T)$ different from $T(v_1)$; for $i = 2, 3, \dots, k$, $T(u_i)$ is the heaviest subtree of u_{i-1} . For each node v_j in π , with $j = 0, 1, \dots, m-1$, (for each node u_i in π , with $i = 1, 2, \dots, k-1$), call *second heaviest subtree* $S(v_j)$ ($S(u_i)$) and *third heaviest subtree* $R(v_j)$ ($R(u_i)$), the subtrees of v_j (of u_i) different from $T(v_{j+1})$ (from $T(u_{i+1})$) with the greater and the smaller number of nodes, respectively.

Recursively construct a drawing satisfying the top visibility property of each subtree of π . Let h be an horizontal grid line. Draw $r(T)$ on h . Place the drawing of $R(v_0)$ with the highest horizontal line intersecting it one unit below h and with its root on the same vertical line of $r(T)$. For $j = 1, 2, \dots, m-1$, rotate of π the drawing of $R(v_j)$. Place the drawing Γ_j^S of $S(v_j)$ and Γ_j^R of $R(v_j)$ so that the highest horizontal line intersecting Γ_j^S is one unit below h , the lowest horizontal line intersecting Γ_j^R is one unit above h , their roots are on the same vertical line, and the leftmost vertical line intersecting Γ_j^S or Γ_j^R is one unit to the right of the rightmost vertical line intersecting Γ_{j-1}^S , Γ_{j-1}^R , or v_{j-1} . Draw v_j on h on the same vertical line of its already drawn children (or draw v_j one unit to the right of the rightmost vertical line intersecting Γ_{j-1}^S , Γ_{j-1}^R , or v_{j-1} if no child of v_j has been drawn). Draw v_m on h one unit to the right of the rightmost vertical line intersecting Γ_{m-1}^S , Γ_{m-1}^R , or v_{m-1} . For path $(u_1, u_2, \dots, u_k)$ and its subtrees, analogously construct a drawing in which the path lies on h , to the left of $r(T)$, and the $S(u_i)$'s and the $R(u_i)$'s are below and above h , respectively (see Fig. 3.c).

It's easy to see that Γ is an SO-drawing satisfying the top visibility property. Let's analyze the area of Γ . Since there is at least one node of T for each vertical grid line intersecting Γ , then its width is $O(n)$. Denote by $H(T)$ the height of the drawing constructed by the algorithm when its input is T . Let also $H(n) = \max\{H(T)\}$ over all ternary trees T with n nodes. Since all subtrees drawn above π (below π) are aligned on their bottom side (on their top side) and since $H(n)$ is a non-decreasing function of n , then $H(n) = H(n_a) + H(n_b) + 1$, where n_a (n_b) is the number of nodes in the heaviest subtree drawn above (below) of π . We claim (1) $n_a + n_b \leq 2n/3$. Namely, we have (2) $n_a \leq n_b$, (3) $n_b \leq n/2$, and (4) $n_a \leq n - 2n_b$. Inequality (2) holds since for each node w in π , $|R(w)| \leq |S(w)|$; inequality (3) follows from the fact that, for each node v_j and u_i in π ,

$|S(v_j)| \leq |T(v_{j+1})|$ and $|S(u_i)| \leq |T(u_{i+1})|$; inequality (4) follows by considering any node v_j (u_i) in π and observing that $|S(v_j)| \leq |T(v_{j+1})|$ ($|S(u_i)| \leq |T(u_{i+1})|$) and that $|R(v_j)| + |S(v_j)| + |T(v_{j+1})| \leq n$ ($|R(u_i)| + |S(u_i)| + |T(u_{i+1})| \leq n$). By (3) we have $n_b = \frac{n}{2} - \alpha$, with $\alpha \geq 0$. If $\alpha \geq n/6$, then by (2) $n_b + n_a \leq 2(n/2 - \alpha) \leq 2n/3$. If $\alpha < n/6$, by (4) we have $n_a \leq n - 2(n/2 - \alpha) = 2\alpha$. Hence $n_b + n_a \leq n/2 - \alpha + 2\alpha = n/2 + \alpha \leq 2n/3$. We claim that $n_a^{(1/\log_2 3)} + n_b^{(1/\log_2 3)} \leq n^{(1/\log_2 3)}$. Hölder's inequality states that (5) $\sum a_i b_i \leq (\sum a_i^p)^{\frac{1}{p}} (\sum b_i^q)^{\frac{1}{q}}$ for every p and q such that $1/p + 1/q = 1$. Substituting into (5) the values $a_1 = n_a^{(1/\log_2 3)}$, $a_2 = n_b^{(1/\log_2 3)}$, $b_1 = b_2 = 1$, $1/p = 1/\log_2 3$, and $1/q = 1 - 1/\log_2 3$, we get $n_a^{(1/\log_2 3)} + n_b^{(1/\log_2 3)} \leq \left[\left(n_a^{(1/\log_2 3)} \right)^{\log_2 3} + \left(n_b^{(1/\log_2 3)} \right)^{\log_2 3} \right]^{(1/\log_2 3)} \cdot [1 + 1]^{(1-1/\log_2 3)} = (n_a + n_b)^{(1/\log_2 3)} \cdot 2^{(1-1/\log_2 3)} \leq (2n/3)^{(1/\log_2 3)} \cdot 2^{(1-1/\log_2 3)} = n^{(1/\log_2 3)} (2^{1/\log_2 3} \cdot 2 \cdot 2^{-1/\log_2 3}) / (3^{1/\log_2 3}) = n^{(1/\log_2 3)} (2/(3^{1/\log_2 3})) = n^{(1/\log_2 3)}$. Hence, $H(n) \leq \max_{n_1+n_2 \leq 2n/3} H(n_1) + H(n_2) + 1$ by induction solves to $H(n) = c \cdot n^{(1/\log_2 3)} - 1$ for some constant c , depending only on the values of $H(n)$ with n small. It follows that $H(n) = O(n^{(1/\log_2 3)}) = O(n^{0.631})$. $\square$

5 Straight-Line Orthogonal Drawings of Complete Ternary Trees

For complete ternary trees we present two algorithms constructing drawings with better area bounds than the ones obtained for general ternary trees. Let Γ_h be a drawing of a complete ternary tree T_h with height h . In both algorithms inductively suppose to have a drawing Γ_{h-1} of T_{h-1} satisfying the top visibility property, take three copies Γ'_{h-1} , Γ''_{h-1} , and Γ'''_{h-1} of Γ_{h-1} , rotate Γ'_{h-1} of $\pi/2$ and Γ''_{h-1} of $3\pi/2$ in clockwise direction. The algorithms differ in the geometric construction of Γ_h . In *Construction 1* draw $r(T_h)$ on any horizontal grid line l_h . Place Γ'''_{h-1} with the highest horizontal line intersecting it one unit below l_h and with the root $r(T_{h-1})$ in Γ'''_{h-1} on the same vertical line of $r(T_h)$. Place Γ'_{h-1} with the rightmost vertical line intersecting it one unit to the left of the leftmost vertical line intersecting Γ'''_{h-1} and with the root $r(T_{h-1})$ in Γ'_{h-1} on l_h . Place Γ''_{h-1} with the leftmost vertical line intersecting it one unit to the right of the rightmost vertical line intersecting Γ'''_{h-1} and with the root $r(T_{h-1})$ in Γ''_{h-1} on l_h (see Fig. 4.a). In *Construction 2* draw $r(T_h)$ on any horizontal grid line l_h . Place Γ'_{h-1} with the rightmost vertical line intersecting it one unit to the left of $r(T_h)$ and with the root $r(T_{h-1})$ in Γ'_{h-1} on l_h . Place Γ'''_{h-1} with the leftmost vertical line intersecting it one unit to the right of $r(T_h)$ and with the root $r(T_{h-1})$ in Γ'''_{h-1} on l_h . Place Γ''_{h-1} with the highest horizontal line intersecting it one unit below the lowest horizontal line intersecting Γ'''_{h-1} , and with the root $r(T_{h-1})$ in Γ''_{h-1} on the same vertical line of $r(T_h)$ (see Fig. 4.b). We have the following:

Theorem 6. *An n -node complete ternary tree T_h admits an $O(n^{1/\log_4 3}) = O(n^{1.262})$ area SO-drawing.*

Fig. 4. Constructions 1 (a) and 2 (b) for SO-drawings of complete ternary trees

Proof: Construct a drawing Γ_h of T_h by inductively using Construction 1. Denoting by W_h and by H_h the width and the height of Γ_h , respectively, by construction we have $W_h = W_{h-1} + 2H_{h-1}$ and $H_h = \max\{W_{h-1}, H_{h-1} + (W_{h-1} + 1)/2\}$. Assume by inductive hypothesis that $W_{h-1} = 2^{h-1} - 1$ and that $H_{h-1} = 2^{h-2}$. Notice that this holds in the base case, where $W_1 = H_1 = 1$. Observe also that by inductive hypothesis $H_{h-1} + (W_{h-1} + 1)/2 = 2^{h-2} + (2^{h-1} - 1 + 1)/2 = 2^{h-1} > W_{h-1} = 2^{h-1} - 1$. Hence, $H_h = 2^{h-1}$ and $W_h = 2^{h-1} - 1 + 2 \cdot 2^{h-2} = 2^h - 1$, that proves the inductive hypothesis. The area of Γ_h is equal to $(2^h - 1) \cdot 2^{h-1} < 4^h = 4^{O(\log_3 n)} = O(n^{1/\log_4 3})$. Inductively applying Construction 2 instead of Construction 1 yields to a drawing with asymptotically the same area. $\square$

Next, we show that n -node complete ternary trees have $\Omega(n^{0.438})$ minimum side in any SO-drawing. This result sharply contrasts with the analogous for binary trees. Namely, *any* binary tree admits an SO-drawing in which one side is $O(\log n)$. Let Γ_h be any SO-drawing of T_h . One of the children of $r(T_h)$, say v_1 , is such that no other child of $r(T_h)$ is drawn on the line l through $r(T_h)$ and v_1 . Moreover, for $i = 1, 2, \dots, h-2$, node v_i has exactly one child v_{i+1} drawn on l . Hence, in any SO-drawing of T_h , there is a spine of h nodes drawn all on the same horizontal or vertical line l , such that no other child of $r(T_h)$ is on l . We call *leg* of Γ_h such a spine. Analogously, in any SO-drawing of T_h there is a double-spine of $2h - 1$ nodes that are drawn all on the same horizontal or vertical line. We call *arm* of Γ_h such a double-spine. We have:

Lemma 2. *The minimum side of any SO-drawing of an n -node complete ternary tree is $\Omega(n^{0.438})$.*

Proof: Let Γ_h be an SO-drawing of a complete ternary tree T_h in which the length of the leg is minimum. Let $l(\Gamma_h)$ be the length of the leg in Γ_h . We claim that $l(\Gamma_h) \geq l(\Gamma_{h-1}) + l(\Gamma_{h-2})$. Consider the arms of the subtrees of $r(T_h)$. Either two of such arms are vertical and one horizontal or vice versa. Assume,

Fig. 5. Illustrations for Lemma 2. Thick lines drawn inside subtrees represent their legs.

possibly rotating Γ_h of $\pi/2$, that two of such arms, say S_1 and S_2 , are vertical and one, say S_3 , horizontal. Consider the possible non-crossing placements of S_1 , S_2 , and S_3 , and consider the lowest horizontal line l intersecting both S_1 and S_2 . Two are the cases; either S_3 is below l (Fig. 5.a), or not (Fig. 5.b). In the first case we trivially have $l(\Gamma_h) > l(\Gamma_{h-1}) + l(\Gamma_{h-2})$ (see Fig. 5.c) and the claim follows. In the second case we have $l(\Gamma_h) > l(\Gamma_{h-1}) + l(\Gamma_{h-2}) + l(\Gamma_{h-5}) + \dots + l(\Gamma_{h/2 - \lfloor h/2 \rfloor + 3}) + l(\Gamma_{h/2 - \lfloor h/2 \rfloor + 1})$ (see Fig. 5.d). However, recurrence (1) $f(x) = f(x-1) + f(x-3) + f(x-5) + \dots + f(x/2 - \lfloor x/2 \rfloor + 3) + f(x/2 - \lfloor x/2 \rfloor + 1)$ asymptotically provides for $f(x)$ the same value provided by $f(x) = f(x-1) + f(x-2)$. Namely, from (1) we get $f(x-2) = f(x-3) + f(x-5) + \dots + f((x-2)/2 - \lfloor (x-2)/2 \rfloor + 3) + f((x-2)/2 - \lfloor (x-2)/2 \rfloor + 1)$, and since $(x-2)/2 - \lfloor (x-2)/2 \rfloor = x/2 - \lfloor x/2 \rfloor$ we get $f(x-3) + f(x-5) + \dots + f(x/2 - \lfloor x/2 \rfloor + 3) + f(x/2 - \lfloor x/2 \rfloor + 1) = f(x-2)$, that substituted in (1) gives $f(x) = f(x-1) + f(x-2)$. Hence, $l(\Gamma_h) \geq l(\Gamma_{h-1}) + l(\Gamma_{h-2})$, implying that $l(\Gamma_h)$ grows at least as the terms of the *Fibonacci series*, for which it is well known that the ratio of two consecutive terms l_{k+1} and l_k tends to the *golden ratio* ϕ . Hence $l(\Gamma_h) = \Omega(\phi^h) = \Omega(\phi^{\log_3 n}) = \Omega(n^{1/\log_3 \phi}) = \Omega(n^{0.438})$. The statement follows by observing that the minimum length of the arm of Γ_h grows asymptotically at least as the leg of Γ_h and that each side of Γ_h is at least long as the leg or as the arm of Γ_h . $\square$

In the following we prove that, for complete ternary trees, the lower bound of Lemma 2 is tight. Again, we introduce two constructions, called Constructions $\hat{1}$ and $\hat{2}$, defined as follows: Construction $\hat{1}$ has the same geometric inductive step of Construction 1, but the side drawings are recursively constructed with Construction $\hat{2}$ and the base drawing is recursively constructed with Construction $\hat{1}$; Construction $\hat{2}$ has the same geometric inductive step of Construction 2, but the side drawings are recursively constructed with Construction $\hat{1}$ and the base drawing is recursively constructed with Construction $\hat{2}$.

Lemma 3. *The drawings built by Construction $\hat{1}$ have $O(n^{0.438})$ height.*

Proof: Denote by H_h^1 (by W_h^2) the height (the width) of the drawing of a complete ternary tree with height h built with Construction $\hat{1}$ (with Construction $\hat{2}$). By simple geometric considerations, we have (1) $H_h^1 = \max\{W_{h-1}^2, H_{h-1}^1 + (W_{h-1}^2 + 1)/2\}$ and (2) $W_h^2 = \max\{W_{h-1}^2, 2H_{h-1}^1 + 1\}$. Suppose by induction that $H_{h-1}^1 + (W_{h-1}^2 + 1)/2 \geq W_{h-1}^2$ and that $2H_{h-1}^1 + 1 \geq W_{h-1}^2$. Such inductive hypotheses are verified in the base case, where $H_1^1 = 1$ and $W_1^2 = 1$. Due to the inductive hypotheses, (1) and (2) turn in (1') $H_h^1 = H_{h-1}^1 + (W_{h-1}^2 + 1)/2$, and (2') $W_h^2 = 2H_{h-1}^1 + 1$, respectively. We have $H_h^1 + (W_h^2 + 1)/2 = H_{h-1}^1 + (W_{h-1}^2 + 1)/2 + (2H_{h-1}^1 + 1 + 1)/2 = 2H_{h-1}^1 + (W_{h-1}^2)/2 + 3/2 > 2H_{h-1}^1 + 1 = W_h^2$, and that $2H_h^1 + 1 = 2H_{h-1}^1 + W_{h-1}^2 + 1 + 1 > 2H_{h-1}^1 + 1 = W_h^2$. Hence, the inductive hypothesis is verified and (1') and (2') hold. Substituting (2') into (1'), we get $H_h^1 = H_{h-1}^1 + ((2H_{h-2}^1 + 1) + 1)/2 = H_{h-1}^1 + H_{h-2}^1 + 1$. As in the proof of Lemma 2, H_h^1 grows as the terms of the Fibonacci series, yielding $H_h^1 = O(n^{0.438})$. $\square$

6 Conclusions and Open Problems

In this paper we have shown some upper bounds (Theorems 2, 4, 5, and 6) and lower bounds (Theorems 1 and 3) concerning the area requirement of straight-line orthogonal drawings of binary and ternary trees. As can be noticed from Table 1 some of these bounds are asymptotically tight, whereas for others there is still space for improvements. In particular, for order-preserving SO-drawings of binary trees and for SO-drawings of ternary trees there are wide gaps between the area upper bounds we provided and the actual lower bounds. For complete ternary trees we conjecture that an algorithm combining Constructions 1 and 2 could improve the upper bound we provided here. However, the most fascinating problem in this area still remains, in our opinion, the one of determining whether binary trees admit straight-line orthogonal drawings in linear area.

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