A Pseudo-Boolean Optimization for Multiple Criteria Decision Making in Complex Systems

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Abstract. In complex system problems, a Decision Maker (DM) is often faced with choosing a subset of alternatives from a bigger set. This process is known as multiple criteria decision making (MCDM). Examples of MCDM include decision making in human resource management, water resource management, environmental management and site selection, energy policy issues, portfolio selection, transportation and routing selection, student admission. In general, there are several criteria that need to be satisfied; however, they are usually at least partly conflicting. In this study, we propose a pseudo-boolean approach for multiple criteria decision making on complex system problems. The computational results illustrate both robustness and attractiveness of this solution approach.

Keywords: Pseudo-Boolean Optimization, Multiple Criteria Decision Making.

1 Introduction

Complex system problems are often hard to model and some of them can be represented by nonlinear objective functions with interconnectedness among multiple dimensional spaces. Conventional mathematical programming approaches are sometimes unable to solve it directly. It is conventional wisdom to apply the combination of artificial intelligent, evolutionary computation, system dynamics and heuristic such as genetic algorithm and ant colony to solve these problems [1,2,3,4,5,6]. Other approaches using chaos theory and morphological analysis are reported in the literature [7,8]. In this study, we apply multiple criteria decision making model to complex systems which can be solved by a pseudo-boolean optimization approach.

pseudo-boolean formulations are widely known for their ability to represent a rich variety of important discrete problems, which are often NP-hard. Hammer and Rudeanu [9] introduced pseudo-boolean optimization in their early work and presented a dynamic programming procedure for solving certain problems.

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In their paper, they gave a definition of a *first-order derivative* and indicated its use in solving discrete pseudo-boolean optimization problems. Detail surveys of pseudo-boolean optimization are given by Boris and Hammer [10] and Crama and Hammer [11].

In this paper we extend this notion of first order derivatives by defining *higher-order derivatives* for discrete pseudo-boolean optimization. Moreover, in the context of changing r elements of x at a time (the so-called *r-flip* moves for $r=2$ and 3), we present closed-form formulas that allow 'efficient' implementation of such compound moves. Then, for the important special cases of quadratic and cubic optimization, we define a general *r-flip* move that allows efficient implementation of multi-exchange neighborhood search process for solving such problems. Finally, we illustrate the use of such moves by applying variants of simple search processes based on *r-flip* moves (with $r = 1$ and 2) to a test bed of complex system problems. The paper then concludes with summary and a look ahead to future research.

2 Multiple Criteria Decision Making

In complex system problems, a Decision Maker (DM) is often faced with choosing a subset of alternatives from a bigger set. This process is known as multiple criteria decision making (MCDM). Examples include decision making in human resource management, water resource management, environmental management and site selection, energy policy issues, portfolio selection, transportation and routing selection, student admission. In complex system, there are several criteria that need to be satisfied; however, they are usually at least partly conflicting. If an alternative is selected in conjunction with other alternatives it has different (positive or negative) effects on a specific criterion. Thus, selected alternatives have *interaction* effects on different criteria. Often a DM in practices uses a simple weighted linear function to rank alternatives then select top few. Not considering *interdependencies* among alternatives can lead to an undesirable outcome. For an illustration consider the following example on Waste Disposal Location (WDL) problem taken from Rajabi et al. [12]. In WDL problem, the objective is to identify the two best among five potential sites of equal capacity. Three criteria for success has been identified, (1) *proximity to population*, (2) *infrastructure requirements*, and (3) *environmental risk*. The WDL problem is characterized by certain interdependencies among alternatives. Building a new road near sites 4 and 5 could serve both sites; if both sites are selected, a saving in infrastructure investment will be obtained. If both sites, 4 and 5, are selected there is an increased of 10% on the infrastructure investment criterion. Similarly, if sites 1 and 2 are simultaneously selected then a single power plant facility may be built for both, taking advantage of economy of scale. In that case, a positive energy of 30% is estimated when it comes to infrastructure investment criterion. Finally, when it comes to environmental risk criterion, selection of site 4 and 5 together has a negative effect of 30% . The scenario is illustrated in the Figure 1, and data is presented in Table 1.

Fig. 1. WDL example: Interdependence actions and synergy levels. (Left: Proximity to population, Middle: Infrastructure requirements, Right: Environmental risk).

Table 1. Normalized consequences of five feasible sites

	Weights	Criteria	Alternatives				
			a^1	a^2	a^3	a^4	a^5
1	0.23	Population	0.45	0.45	1.00	0.55	0.84
2	0.39	Infrastructure	0.80	0.70	0.75	0.83	0.83
3	0.38	Environmental Risk	0.60	0.87	0.50	0.75	0.60
Additive Value			0.644	0.707	0.713	0.735	0.745

The following notations are adopted in the paper:
 $N = \{a^1, \dots, a^n\}$ is set of alternatives.
 $a^i, (i = 1, \dots, n)$ is alternative i .
 $P = \{1, \dots, n\}$ is set of criteria.
 c_k^i is effect of alternative i on criterion k .
 S_k is a set of alternatives that if selected together have some positive or negative effect on criterion k .
 $\gamma(S_k)$ is the amount of effect (positive or negative) of an *interacting* set S_k on criterion k .
 w_k is weight associated with criterion k .
 $v(S)$ is total payoff of a subset of alternatives $S \subseteq N$.
 x_i is equal to 1 if an alternative a^i is selected and 0 otherwise.

To illustrate the notations, consider Table 1, we have 5 alternatives $N = \{1, 2, 3, 4, 5\}$, and 3 criteria $P = \{1, 2, 3\}$. Alternative A^2 has effects of $c_1^2 = 0.45, c_2^2 = 0.70$, and $c_3^2 = 0.87$ on criterion, 1, 2, and 3, respectively. Weights of criteria are $w = (w_1, w_2, w_3) = (0.23, 0.39, 0.38)$. Considering criterion 2 there are two interacting alternatives, $S_2 = \{a^1, a^2\}$ and $S_2 = \{a^4, a^5\}$, with interaction effects (positive energy) of $\gamma(\{a^1, a^2\}) = 0.30$ and $\gamma(\{a^4, a^5\}) = 0.10$. If no

interdependencies are considered among the alternatives then the payoff value is defined as a simple additive function, for example, for $S = \{a^i, a^j\}$ we have

$$v(a^i, a^j) = v(a^i) + v(a^j) = \sum_{k=1}^3 w_k c_k^i + \sum_{k=1}^3 w_k c_k^j.$$

In this case, considering all possible set of 2 alternatives and selecting a subset with the largest payoff we have optimal solution as $v(a^4, a^5) = v(a^4) + v(a^5) = 0.735 + 0.745 = 1.48$. However, this is a simplistic view of the situation. If we consider *interdependencies* between alternatives and taking into accounts positive and negative energies we have a *dynamic weight* for each criterion when applied to an alternative. The dynamic weight of an alternative depends on its interaction with other alternatives in the selected subset.

3 Model and Solution Methodology

In the following, we will present a pseudo-boolean formulation of the problem that can overcome both of these deficiencies if appropriate solution procedure is available. The pseudo-boolean optimization for the problem can be presented as a maximization problem given below [10].

$$\begin{aligned} \text{Max} \quad & \sum_{i \in N} x_i \left[\sum_{k \in P} w_k c_k^i + \sum_{k \in P} \sum_{s_k \subseteq N} \left(w_k [\gamma(S_k)] \left[\sum_{i \in S_k} c_k^i \right] \right) \right] \prod_{i \in s_k} x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = m, \\ & x_i \in \{0, 1\}. \end{aligned}$$

In this problem, if we want to find any subset of alternatives with largest payoff, then we remove the equality constraint and solve the problem.

To illustrate the optimization problem we use the WDL example presented below.

$$\begin{aligned} \text{Max} \quad & .644x_1 + .707x_2 + .713x_3 + .735x_4 + .745x_5 + .39(.30)[.80 + .70]x_1x_2 + \\ & .39(.10)[.83 + .83]x_4x_5 - .39(.30)[.75 + .60]x_4x_5 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_5 = 2, \\ & x_i \in \{0, 1\}. \end{aligned}$$

Simplifying the above formulation we get

$$\begin{aligned} \text{Max} \quad & .644x_1 + .707x_2 + .713x_3 + .735x_4 + .745x_5 + .1755x_1x_2 - .8916x_4x_5 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_5 = 2, \\ & x_i \in \{0, 1\}. \end{aligned}$$

Glover [13] used an efficient *1-flip* search implementation and tested it in the context of Tabu Search and Scatter Search.

Definition 1. Let $x = (x_1, \dots, x_i, \dots, x_n)$ be a solution to pseudo-boolean and let $\bar{x} = (x_1, \dots, \bar{x}_i, \dots, x_n)$ be a solution obtained from x by complementing x_i .

Since Q is upper triangular matrix then (ij) th element, q_{ij} , is equal to the second derivative of $f(x)$, i.e., $\delta_{ij}(x) = q_{ij}$, with respect to x_i and x_j .

Theorem 1 (Glover et al.[14]). For x and x' as given in Definition 1, the change in the value of the objective function is

$$f(x') - f(x) = (x'_i - x_i)\Delta_i(x)$$

Moreover, in any local optimal solution of the pseudo-boolean with respect to 1-flip search we have

$$\text{either } (x_i = 0 \iff \Delta_i(x) \geq 0) \text{ or } (x_i = 1 \iff \Delta_i(x) \leq 0), \forall i = 1, \dots, n.$$

Furthermore, after changing x_i to $\bar{x}_i$ the update for all $\Delta_i(x)$ can be calculated as follows

$$\begin{aligned} \forall j < i, & \quad \Delta_j(x) = \Delta_j(x) + q_{ji}(x'_i - x_i), \\ \forall j > i, & \quad \Delta_j(x) = \Delta_j(x) + q_{ij}(x'_i - x_i), \\ j = i, & \quad \Delta_j(x) = \Delta_i(x). \end{aligned}$$

In Theorem 2, we extend the above result to general case for r -flip search for pseudo-boolean.

Theorem 2. Let x be a given solution of pseudo-boolean and x' obtained from x by an r -flip move where $S \subseteq V$, and $|S| = r$. Now, the change in the value of the objective function is

$$f(x') - f(x) = \sum_{i \in S} (\bar{x}_i - x_i)\Delta_i(x) + \sum_{\forall i, j \in S} (\bar{x}_i - x_i)(\bar{x}_j - x_j)\Delta_{ij}(x).$$

Furthermore, after changing x to x' the update for all $\Delta_i(x)$ can be calculated as follows,

$$\begin{aligned} \forall j \in V \setminus S & \quad \Delta_j(x) = \Delta_j(x) + \sum_{i \in S} (\bar{x}_i - x_i)\Delta_{ij}(x), \text{ and} \\ \forall j \in S & \quad \Delta_j(x) = \Delta_j(x) + \sum_{i \in S \setminus \{j\}} (\bar{x}_i - x_i)\Delta_{ij}(x). \end{aligned}$$

Proof. For two vectors x' and x where component x_i for all $i \in S$ are complemented we have $f(x)$ and $f(x')$ as follows:

$$f(x) = \sum_{i \in S} x_i[q_{ij} + \sum_{j \notin S} x_j q_{ij}] + \sum_{ij \in S} x_i x_j q_{ij} + \dots$$

$$f(x) = \sum_{i \in S} x_i[\Delta_i(x) - \sum_{j \notin S \setminus \{i\}} x_j q_{ij}] + \sum_{ij \in S} x_i x_j q_{ij} + \dots$$

$$f(x') = \sum_{i \in S} \bar{x}_i[q_{ij} + \sum_{j \notin S} x_j q_{ij}] + \sum_{ij \in S} \bar{x}_i \bar{x}_j q_{ij} + \dots$$

$$f(x') = \sum_{i \in S} \bar{x}_i[\Delta_i(x) - \sum_{j \notin S \setminus \{i\}} x_j q_{ij}] + \sum_{ij \in S} \bar{x}_i \bar{x}_j q_{ij} + \dots$$

$$f(x') - f(x) = \sum_{i \in S} (\bar{x}_i - x_i)[\Delta_i(x) - \sum_{j \notin S \setminus \{i\}} x_j q_{ij}] + \sum_{ij \in S} (\bar{x}_i \bar{x}_j - x_i x_j) q_{ij}$$

$$f(x') - f(x) = \sum_{i \in S} (\bar{x}_i - x_i)\Delta_i(x) - \sum_{j \in S} \sum_{j \notin S \setminus \{i\}} (\bar{x}_i - x_i)x_i q_{ij} + \sum_{ij \in S} (\bar{x}_i \bar{x}_j - x_i x_j) q_{ij}$$

For each pair of elements i, j in the second and the third terms we have
 $\bar{x}_i \bar{x}_j + x_i x_j - \bar{x}_i x_j - x_i \bar{x}_j = (\bar{x}_i - x_i)(\bar{x}_j - x_j)$

From this we get

$$f(x') - f(x) = \sum_{i \in S} (\bar{x}_i - x_i) \Delta_i(x) + \sum_{\forall i, j \in S} (\bar{x}_i - x_i)(\bar{x}_j - x_j) \Delta_{ij}(x)$$

which proves the first part of the theorem.

Note that we can substitute $\Delta_{ij}(x)$ for q_{ij} . From this the desired result follows immediately. We propose several methods to solve the transformed pseudo-boolean problem.

Filtration and Sequential Fan(F&F) (see Glover[13]) is a search process utilizing compound move strategies that have proven to be effective for a variety of combinatorial problems. Here we employ a simple version of F&F utilizing a combination of r -flip moves. In the paragraph below we give a brief overview of our implementation.

F&F organizes an aggressive search process utilizing a variety of different r -flip moves. The version we tested here starts with a set of locally optimal solutions found using 1 -flip search. Then, for each solution in this set a second round of 1 -flip moves is executed. If an improving move is found we accept that move and initiate a complete new round of 1 -flip moves. If an improving second flip move can not be found, we initiate a series of 3 -flip moves starting with the best local optimal solutions found so far. This continues until a pre-set stopping criterion is satisfied or until no further improvement is realized.

4 Computational Results

Table 2 presents all subset of 2 alternatives and associated weights of each criterion, and payoff value of the selected subset of alternatives. In that, obviously subset (a^4, a^5) is not any more the optimal solution as we have $v(a^4, a^5) = 1.391$. However, the subset (a^1, a^2) has the largest payoff of $v(a^1, a^2) = 1.526$ which is optimal. Table 3 illustrates calculation of weights for each criterion for all subsets of 4 alternatives solved via Method 4. The dynamic nature of weights is clear from these tables. Obviously, when the interdependencies among

Table 2. Calculation of payoff when interdependencies are considered

2 alternatives (a^i, a^j)	w_1	w_2	w_3	$v(a^i, a^j)$
1,2	0.23	$0.39(1+0.30)=0.507$	0.38	1.526
1,3	0.23	0.39	0.38	1.356
1,4	0.23	0.39	0.38	1.379
1,5	0.23	0.39	0.38	1.388
2,3	0.23	0.39	0.38	1.420
2,4	0.23	0.39	0.38	1.442
2,5	0.23	0.39	0.38	1.452
3,4	0.23	0.39	0.38	1.448
3,5	0.23	0.39	0.38	1.457
4,5	0.23	$0.39(1+0.10)=0.429$	$0.38(1-0.30)=0.266$	1.391

Table 3. Calculation of weights for subset of 4 alternatives when interdependencies are considered

	a^1	a^2	a^3	a^4	a^5
w_1	0.23	0.23	0.23	0.23	
w_2	$0.39(1+0.30)=0.507$	$0.39(1+0.30)=0.507$	0.39	0.39	
w_3	0.38	0.38	0.38	0.38	
w_1	0.23	0.23	0.23		0.23
w_2	$0.39(1+0.30)=0.507$	$0.39(1+0.30)=0.507$	0.39		0.39
w_3	0.38	0.38	0.38		0.38
w_1	0.23	0.23		0.23	0.23
w_2	$0.39(1+0.30)=0.507$	$0.39(1+0.30)=0.507$		$0.39(1+0.10)=0.429$	$0.39(1+0.10)=0.429$
w_3	0.38	0.38		$0.38(1-0.30)=0.266$	$0.38(1-0.30)=0.266$
w_1	0.23		0.23	0.23	0.23
w_2	0.39		0.39	$0.39(1+0.10)=0.429$	$0.39(1+0.10)=0.429$
w_3	0.38		0.38	$0.38(1-0.30)=0.266$	$0.38(1-0.30)=0.266$
w_1		0.23	0.23	0.23	0.23
w_2		0.39	0.39	$0.39(1+0.10)=0.429$	$0.39(1+0.10)=0.429$
w_3		0.38	0.38	$0.38(1-0.30)=0.266$	$0.38(1-0.30)=0.266$

alternatives on different criteria are high the level of *dynamism* for each weight is also high. While considering the interdependencies among alternatives represents the practical nature of the problem, however calculating the dynamic weights is very difficult. If we need to select m alternatives with highest total payoff we have to consider calculation of weights in all possible m set of alternatives. And, in case, the problem is to select the best subset of alternatives (with no pre-specified number of elements) we have to calculate weights for all subset of alternatives. In both cases, the number of subsets is exponential.

5 Summary and Conclusion

In this paper we reported on modeling complex system problems with the pseudo-boolean formulation and proposed several solution approaches. The complex systems with very high-dimensional spaces often associate with more complicated variables. The pseudo-boolean formulation can reduce the variable dimension without loss important information. The preliminary results illustrate the attractiveness of this modeling method in term of generating the best alternative. In our continuing work, we intend to carry out further testing of pseudo-boolean formulation on cubic, quartic, even quintic variables.

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