The Equitable Edge-Coloring of Series-Parallel Graphs*

Huimin Song¹, Jianliang Wu², and Guizhen Liu²

¹ Department of Applied Mathematics, Shandong University at Weihai, Weihai, 264209, P.R. China

² School of Mathematics and System Science, Shandong University, Jinan, 250100, P.R. China

Abstract. An edge-coloring of a graph G is *equitable* if, for each vertex v of G , the number of edges of any one color incident with v differs from the number of edges of any other color incident with v by at most one. A graph G is called equitable if G has an equitable edge-coloring with k colors for any integer $k \geq 1$. A plane graph is series-parallel graph if it contains no subgraphs homeomorphic to K_4 . In the paper, we prove that any simple and connected series-parallel graph is equitable if and only if it is not an odd circuit.

Keywords: edge-coloring; equitable, series-parallel graph.

1 Introduction

Throughout the paper, all graphs are finite, simple and undirected. Let G be a graph, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G , respectively. We associate positive integer $1, 2, \dots$ with colors, and call C a k -edge-coloring of G if $C : E \rightarrow \{1, 2, \dots, k\}$. Let $C_v^{-1}(i)$ denote the number of edges of G incident with vertex v that receive color i by the coloring C . We call C an equitable k -edge-coloring of G if for each vertex $v \in V$

$$|C_v^{-1}(i) - C_v^{-1}(j)| \leq 1 \quad (1 \leq i < j \leq k).$$

A graph G is called *equitable* if G has an equitable k -edge-coloring for any integer $k \geq 1$. It is an interesting problem to determine whether a graph is equitable or not. If a graph is equitable, then its edge chromatic index is equal to its maximum degree Δ and its edge cover chromatic index equal to its minimum degree δ (refer to [1] for the definition of edge cover chromatic index).

A plane graph is series-parallel graph if it contains no subgraphs homeomorphic to K_4 , denoted by SP graph briefly. A series-parallel graph is a kind of very interesting graph which has many good properties and is well applied in electricity networks.

A circuit is a connected graph in which each vertex has even degree. A circuit is odd, or even, if the number of edges is odd, or even, respectively. It is stated in

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[2] that a connected graph G has an equitable 2-edge-coloring if and only if it is not an odd circuit. Hilton and Werra [3] gave a sufficient condition for equitable edge-coloring of simple graphs. Werra[2] has proved that all bipartite graphs are equitable. Wu Jian-liang [4] has proved that a connected outerplanar graph is equitable if and only if it is not an odd circuit.

2 The Main Result

In this paper, we consider the equitable edge-coloring of simple series-parallel graphs and obtain the following result.

Theorem 1. *Any simple and connected series-parallel graph is equitable if and only if it is not an odd circuit.*

Firstly we discuss the properties on the structure of series-parallel graphs. Wu Jian-liang has made many researchs on the structure of series-parallel graphs and gained some results by which some problems on the list edge-coloring and the linear arboricity of series-parallel graphs are successfully solved (see [5,6]). We will discuss the structure of series-parallel graphs more meticulously by new methods. Our result on the structure of series-parallel graphs will be more appropriate for the discussion of the equitable edge-coloring of series-parallel graphs.

A 2-connected series-parallel graph can be gained from K_2 by iteratively using the following operations: Replace an edge with a path of length 2 (denoted by S-operation briefly) or duplicate an edge (denoted by P-operation briefly). For every 2-connected series-parallel graph, we can express the operation process with a complete binary tree. For we always begin constructing a SP graph with K_2 , let an isolated point denote the only edge in K_2 at the beginning. If an S-operation is performed on an edge, call the point representing the edge S-node and join two son nodes to the S-node each of which represents one of the two edges on the path of length 2 respectively; If a P-operation is performed on an edge, call the point representing the edge P-node and join two son nodes to the P-node which represent the edge and its duplicate respectively. By the above process, every 2-connected series-parallel graph G corresponds to a complete binary tree T .

By discussing the corresponding complete binary trees in detail, we obtain the properties on the structure of simple SP graphs which are described in the following lemma. In the following, $N_G(v)$ denotes the set of vertices adjacent to a vertex v in G , without confusion, $N(v)$ briefly. A vertex v is called a k -vertex if $|N_G(v)| = k$.

Lemma 1. *Let G be a simple and 2-connected series-parallel graph of order at least 5. Then one of the following conditions holds:*

1. G has two adjacent 2-vertices x and y ;
2. G has two different 2-vertices x and y and $N(x) = N(y)$;
3. G has a 4-vertex z adjacent to two 2-vertices x and y such that $N(z) \setminus \{x, y\} = \{N(x) \cup N(y)\} \setminus \{z\}$;

4. G has a 3-vertex w with $N(w) = \{x, y, z\}$ such that both x and y are 2-vertices, $N(x) = \{z, w\}$ and $yz \notin E(G)$;
5. G has two adjacent 3-vertices x and y such that $N(x) \cap N(y) = \{z\}$ and $N(z) = \{x, y\}$;
6. G has two adjacent 3-vertices w_1 and w_2 such that $N(w_1) = \{x, z_1, w_2\}$, $N(w_2) = \{y, z_2, w_1\}$, $N(x) = \{z_1, w_1\}$ and $N(y) = \{z_2, w_2\}$;
7. G has a 3-vertex w with $N(w) = \{x, y, z\}$ such that $N(z) = \{w, y\}$ and $xy \in E(G)$;
8. G has two nonadjacent 3-vertices w_1 and w_2 such that $N(w_1) = \{x, y, z_1\}$, $N(w_2) = \{x, y, z_2\}$, $N(z_1) = \{x, w_1\}$ and $N(z_2) = \{y, w_2\}$;
9. G has two nonadjacent 3-vertices w_1 and w_2 such that $N(w_1) = \{x, y, z_1\}$, $N(w_2) = \{x, y, z_2\}$, $N(z_1) = \{x, w_1\}$ and $N(z_2) = \{x, w_2\}$;
10. G has a 3-vertex w with $N(w) = \{x, z_1, z_2\}$ such that there is a 2-vertex $y \in N(z_1) \cap N(z_2)$ and $N(x) = \{z_1, w\}$.

Let C denote color set $\{1, 2, \dots, k\}$. Given a edge-coloring φ of G in which k colors in C are used. For each vertex $v \in V(G)$, let $\varphi_v^{-1}(i)$ denote the number of edges of G incident with vertex v that receive color i by the coloring φ , and let $C_\varphi(v) = \{i | \varphi_v^{-1}(i) = \min_{1 \leq j \leq k} \varphi_v^{-1}(j)\}$. Obviously, $|C_\varphi(v)| \geq 1$.

Now we will give the main idea for the proof of our theorem. It is known that a connected graph G has an equitable 2-edge-coloring if and only if it is not an odd circuit. So it is only to prove that any simple and connected SP graph G has an equitable k -edge-coloring for any integer $k \geq 3$. We prove the result by induction on $|G|$, the number of vertices of G .

When $|G| \leq 4$, the result is obvious. Assume that p ($p \geq 5$) is an integer and that the theorem holds for all simple SP graphs with less than p vertices. Let G be a simple and connected SP graph of order p . If G is not 2-connected, there is a cut vertex $v \in V(G)$ such that $G - v$ is not connected. Select a component of $G - v$ arbitrarily, let S denote its vertex set, $\bar{S}$ denote the set of vertices in $G - v$ which do not belong to S . Let G_1 denote the subgraph of G induced by $S \cup \{v\}$, G_2 denote the subgraph of G induced by $\bar{S} \cup \{v\}$. It is obvious that $E(G) = E(G_1) \cup E(G_2)$ and $V(G) = V(G_1) \cup V(G_2)$. Because both G_1 and G_2 are SP graphs of order less than p , then G_1 and G_2 have equitable k -edge-colorings φ_1 and φ_2 respectively. Let $k_1 = k - |C_{\varphi_1}(v)|$, $k_2 = |C_{\varphi_2}(v)|$. By interchanging the colors used, an equitable k -edge-coloring φ'_1 of G_1 and an equitable k -edge-coloring φ'_2 of G_2 will be gained such that $C \setminus C_{\varphi'_1}(v) = \{1, 2, \dots, k_1\}$, $C_{\varphi'_2}(v) = \{1, 2, \dots, k_2\}$. Combine coloring φ'_1 in G_1 with coloring φ'_2 in G_2 , an equitable k -edge-coloring of G will be gained.

In the following we suppose that G is 2-connected. According to Lemma 1, the proof can be divided into ten parts. In every case as described in Lemma 1, we always obtain a simple SP graph G^* of order less than p . By the induction hypothesis, G^* has an equitable k -edge-coloring φ . On the basis of φ , we always can construct an equitable k -edge-coloring σ of G using the same set of k colors. By the induction, the theorem is true.

3 Concluding Remarks

An outerplanar graph is a graph which contains no subgraphs homeomorphic to K_4 or $K_{2,3}$. That is to say, an outerplanar graph is also a series-parallel graph. From this point of view, our result generalizes the result in [4].

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