On f -Edge Cover-Coloring of Simple Graphs^{*}

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Abstract. Let $G(V, E)$ be a simple graph, and let f be an integer function on V with $1 \leq f(v) \leq d(v)$ to each vertex $v \in V$. An f -edge cover-coloring of a graph G is a coloring of edge set E such that each color appears at each vertex $v \in V$ at least $f(v)$ times. The f -edge cover chromatic index of G , denoted by $\chi'_{fc}(G)$, is the maximum number of colors such that an f -edge cover-coloring of G exists. Any simple graph G has f -edge cover chromatic index equal to δ_f or $\delta_f - 1$, where $\delta_f = \min_{v \in V} \{\lfloor \frac{d(v)}{f(v)} \rfloor\}$. If $\chi'_{fc}(G) = \delta_f$, then G is of C_f I class; otherwise G is of C_f II class. In this paper, we give some sufficient conditions for a graph to be of C_f I class, and discuss the classification problem of complete graphs on f -edge cover-coloring.

Keywords: Simple graph, Edge-coloring, f -edge cover-coloring.

1 Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. Graphs in this paper are simple, unless otherwise stated, i.e., they have no loops or multiple edges. We use V and E to denote, respectively, the vertex set and edge set of a graph G . Let $N_G(v)$ denote the neighborhood of v and let $d(v) = |N_G(v)|$ be the degree of v in G . We write δ for the minimum degree of G . We associate positive integer $1, 2, \dots$ with colors, and we call C a k -edge coloring of G if $C : E \rightarrow \{1, 2, \dots, k\}$. Let $C_i(v)$ denote the number of edges of G incident with vertex v that receive color i by the coloring C . Assume that a positive integer $f(v)$ with $1 \leq f(v) \leq d(v)$ is associated with each vertex $v \in V$. We call C an f -edge cover-coloring of G if for each vertex $v \in V$, $C_i(v) \geq f(v)$ for $i = 1, 2, \dots, k$, that is each color appears at each vertex $v \in V$ at least $f(v)$ times. Let $\chi'_{fc}(G)$ denote the maximum number of colors for which an f -edge cover-coloring of G exists. We call $\chi'_{fc}(G)$ the f -edge cover chromatic index of G . In our daily life many problems on optimization and network design, e.g., coding design, the file transfer problem on computer networks, schedule problems and

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so on, are related to the f -edge cover-coloring which was first presented by Song and Liu [5].

Let $\delta_f = \min_{v \in V} \{\lfloor \frac{d(v)}{f(v)} \rfloor\}$, in which $\lfloor \frac{d(v)}{f(v)} \rfloor$ is the largest integer not larger than $\frac{d(v)}{f(v)}$. It is easy to verify that $\chi'_{fc}(G) \leq \delta_f$. Song and Liu studied the bound of f -edge cover chromatic index of a graph and obtained the following result.

Theorem 1. [5] *Let G be a graph. Let $f(v)$, $d(v)$ and δ_f be as defined earlier. Then*

$$\delta_f - 1 \leq \min_{v \in V} \{\lfloor \frac{d(v) - 1}{f(v)} \rfloor\} \leq \chi'_{fc}(G) \leq \delta_f.$$

From the above theorem we can see that the f -edge cover chromatic index of any graph G must be equal to δ_f or $\delta_f - 1$. This immediately gives us a simple way of classifying graphs into two types according to $\chi'_{fc}(G)$. More precisely, we say that G is of C_f I class if $\chi'_{fc}(G) = \delta_f$, and that G is of C_f II class if $\chi'_{fc}(G) = \delta_f - 1$.

Song and Liu also obtained the following two results.

Theorem 2. [5] *Let G be a bipartite graph. Let $f(v)$ and δ_f be as defined earlier. Then $\chi'_{fc}(G) = \delta_f$.*

Theorem 3. [5] *Let G be a graph. Let $f(v)$ and δ_f be as defined earlier. If $f(v)$ is positive and even for all $v \in V$, then $\chi'_{fc}(G) = \delta_f$.*

If $f(v) = 1$ for all $v \in V$, the classification problem on f -edge cover-coloring is reduced to the classification problem on the ordinary edge cover-coloring which was studied by Miao [4] and Wang [6]. Hilton [2] studied the f -edge cover-coloring of multigraph G with $f(v) = j \geq 2$ for all $v \in V$. In this paper, we study the f -edge cover chromatic index of simple graphs. In section 2, we give some sufficient conditions for a simple graph to be of C_f I class. In section 3, we discuss the classification problem of complete graphs on f -edge cover-coloring.

2 The Classification of Graphs on f -Edge Cover-Coloring

In this section, we give some sufficient conditions for a graph to be of C_f I class.

Let $V^* = \{v : \delta_f = \lfloor \frac{d(v)}{f(v)} \rfloor, v \in V\}$, then we have the following important theorem.

Theorem 4. *Let G be a graph. Let $f(v)$ and V^* be as defined earlier. If $f(v^*) \nmid d(v^*)$ for all $v^* \in V^*$, then G is of C_f I class.*

Proof. By Theorem 1, it is enough to prove that $\min_{v \in V} \{\lfloor \frac{d(v)-1}{f(v)} \rfloor\} \geq \delta_f$. For $\delta_f = \min_{v \in V} \{\lfloor \frac{d(v)}{f(v)} \rfloor\}$, we have $\lfloor \frac{d(v)}{f(v)} \rfloor > \delta_f$ for each vertex $v \in V \setminus V^*$. So $\lfloor \frac{d(v)-1}{f(v)} \rfloor \geq \delta_f$.

On the other hand, for each vertex $v^* \in V^*$, $f(v^*) \nmid d(v^*)$ implies that $\frac{d(v^*)}{f(v^*)} > \lfloor \frac{d(v^*)}{f(v^*)} \rfloor = \delta_f$. So $\frac{d(v^*)-1}{f(v^*)} \geq \lfloor \frac{d(v^*)}{f(v^*)} \rfloor = \delta_f$, then we have $\min_{v \in V} \{\lfloor \frac{d(v)-1}{f(v)} \rfloor\} \geq \delta_f$. Thus $\chi'_{fc}(G) = \delta_f$, this completing the proof. $\square$

Let k be a positive integer and the k -core of G be the subgraph of G induced by the vertices v of G such that $k \mid d(v)$. The following lemma is given by Hilton and de Werra.

Lemma 1. [3] *Let G be a graph and let $k \geq 2$. If the k -core of G is a set of isolated vertices, then G has a k -edge coloring such that for each vertex $v \in V$ and each $i, j \in \{1, \dots, k\}$, $|C_i(v) - C_j(v)| \leq 1$ holds.*

Theorem 5. *Let G be a graph. Let $f(v)$, $d(v)$, δ_f and V_0^* be as defined earlier. If $V_0^* = \{v : \delta_f = \frac{d(v)}{f(v)}, v \in V\}$ is a set of isolated vertices in graph G , then G is of C_f I class.*

Proof. If $\delta_f = 1$, obviously, G is of C_f I class. Assume that $\delta_f \geq 2$. Let $V_0 = \{v : \delta_f \nmid d(v), v \in V\}$ and $S = V_0 \setminus V_0^*$. It is easy to prove that for each vertex $v \in S$, $\frac{d(v)}{f(v)} > \delta_f$, that is $\frac{d(v)}{\delta_f} > f(v)$. Since $f(v)$ and δ_f are both integer, and $\delta_f \mid d(v)$. Then $\frac{d(v)}{\delta_f} \geq f(v) + 1$.

We shall form graph G' by adding a pendant edge to each vertex of S on graph G . Clearly, $\delta_f \nmid d(v)$ for each vertex $v \in S$, and V_0^* is the δ_f -core of G' which is a set of isolated vertices. By Lemma 1, G' has a δ_f -edge coloring such that for each vertex $v \in V(G)$ and $\forall i, j \in \{1, \dots, \delta_f\}$, $|C_i(v) - C_j(v)| \leq 1$ holds. Since for each vertex $v \in S$, we notice that $\lfloor \frac{d_{G'}(v)}{\delta_f} \rfloor \geq \lfloor \frac{d(v)}{\delta_f} \rfloor \geq f(v) + 1$, and $|C_i(v) - C_j(v)| \leq 1$ ($\forall i, j \in \{1, \dots, \delta_f\}$). That is, every color of $\{1, 2, \dots, \delta_f\}$ is represented at least $f(v) + 1$ times at v . By restricting the δ_f -edge coloring of G' to G , we obtain an f -edge cover-coloring of G with δ_f colors. That is, G is of C_f I class. This proves the Theorem. $\square$

3 The f -Edge Cover-Coloring of Complete Graphs

In this section, we discuss the classification problem of complete graphs on f -edge cover-coloring. Our main result is Theorem 6, in order to prove it, we need the following useful lemma which appears in [3].

Lemma 2. [3] *Let C be a k -edge coloring of graph G . If $|C_\alpha(v_0) - C_\beta(v_0)| > 2$ for some pair α, β of colors and some vertex v_0 . Then G must be recolored with k colors such that*

$$\max_{1 \leq i < j \leq \delta} |C'_i(v_0) - C'_j(v_0)| \leq 2.$$

And for each $v \in V$, the number of colors appears on vertex v would not reduce.

A k -factor of a graph G is a k -regular spanning subgraph. Let G be an mk -regular graph. If $H_1, H_2, \dots, H_m$ are edge-disjoint k -factors of G , then which is called a k -factorization of G .

Theorem 6. *Let G be a complete graph K_n . If k and n are odd integers, $f(v) = k$ and $k \mid d(v)$ for all $v \in V$, then G is of C_f II class. Otherwise, G is of C_f I class.*

Proof. If $\delta_f = 1$, obviously, G is of C_f I class. Assume that $\delta_f \geq 2$. Let $f^* = \max_{v \in V} \{f(v)\}$. If $f^* \nmid (n-1)$, then by Theorem 4, G is of C_f I class. Next suppose $f^* \mid (n-1)$, we consider the following two cases.

Case 1: n is even, suppose $n = 2t$, where t is a positive integer. Then $f^* \mid (2t-1)$. It is well-known that G has a 1-factorization, and we can give an f -edge cover-coloring of G with δ_f colors. G is of C_f I class.

Case 2: n is odd. Let $n = 2t + 1$, where t is a positive integer. There are three subcases.

Subcase 2.1. f^* is even, let $f^* = 2m$. It is well-known that $G = K_{2t+1}$ has a 2-factorization, and we also can give an f -edge cover-coloring of G with δ_f colors. So G is of C_f I class.

Subcase 2.2. If $f(v) = k$, $k \mid 2t$ and k is odd for all $v \in V$. We show that G is of C_f II class. Because $\delta_f = \min_{v \in V} \{\lfloor \frac{d(v)}{f(v)} \rfloor\}$, we have $\delta_f = \frac{d(v)}{k} = \frac{2t}{k}$. Suppose that $G = K_{2t+1}$ is of C_f I class. Thus $c_i(v) = k$ for each $i \in \{1, 2, \dots, \delta_f\}$ and each $v \in V$. The number of edges colored with a color of δ_f colors is $\frac{k(2t+1)}{2}$. Obviously, $\frac{k(2t+1)}{2}$ is not an integer. This contradicts to our assumption. So $G = K_{2t+1}$ is of C_f II class when $f(v) = k$ is odd and $k \mid d(v)$ for all $v \in V$.

Subcase 2.3. f^* is odd and there exists at least a vertex $v \in V$ such that $f(v) < f^*$. Clearly, $\delta_f = \frac{2t}{f^*}$. We will give an f -edge cover-coloring of G with δ_f colors as follows. Suppose v' is the vertex of graph G such that $f(v') \leq f^* - 1$. Let $G' = G \setminus \{v'\}$. Obviously, $G' = K_{2t}$, and let $H = \{H_1, H_2, \dots, H_{2t-1}\}$ is a 1-factorization of G' . Partition H into δ_f subsets $A_1, A_2, \dots, A_{\delta_f}$, in which $|A_{\delta_f}| = f^* - 1$ and $|A_i| = f^*$, $1 \leq i \leq \delta_f - 1$. Assign color i to the edges in A_i , for each $i \in \{1, 2, \dots, \delta_f\}$. Clearly, $C_i(v) = f^*$ for all $v \in V(G')$ and all $i \in \{1, 2, \dots, \delta_f - 1\}$, but $C_{\delta_f}(v) = f^* - 1$ for all $v \in V(G')$. Let $E_{v'} = \{v'u : u \in V(G')\}$, and we color the edges in $E_{v'}$ with color δ_f . Then $C_i(v) = f^* \geq f(v)$ for all $v \in V(G')$ and all $i \in \{1, 2, \dots, \delta_f\}$, and $C_{\delta_f}(v') - C_i(v') = 2t$ for all $i \neq \delta_f$. We apply Lemma 2 successively to vertex v' for each color of $\{1, 2, \dots, \delta_f - 1\}$ and color δ_f , until we ultimately obtained a δ_f -edge coloring C' of graph G such that for each vertex $v \neq v'$ and for all $i \in \{1, \dots, \delta_f\}$, $C'_i(v) = C_i(v) = f^* \geq f(v)$ and $|C'_i(v') - C'_j(v')| \leq 2$ for all $i, j \in \{1, \dots, \delta_f\}$. It implies $C'_i(v') \geq \lfloor \frac{2t}{\delta_f} \rfloor - 1 = f^* - 1 \geq f(v')$ for each $i \in \{1, \dots, \delta_f\}$. Then C' is an f -edge cover-coloring of G . Graph G is of C_f I class. This completes the proof of the theorem. $\square$

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