The Edge Connectivity of Circuit Graphs of Matroids^{*}

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Abstract. Let G be the circuit graph of any connected matroid and let $\kappa'(G)$ and $\delta(G)$ be the edge connectivity and minimum degree of G . It is proved that $\kappa'(G) = \delta(G)$.

Keywords: Matroid, circuit graph of matroid, edge connectivity.

1 Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see [2] for graphs, and [8] for matroids. Let $\mathcal{C}$ be the family of circuits of a matroid M . It satisfies the following two axioms:

(C1) A proper subset of a member of $\mathcal{C}$ is not a member of $\mathcal{C}$.

(C2) If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$ where $C_1, C_2 \in \mathcal{C}$ and $a, b \in E$, then there exists a $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq (C_1 \cup C_2) - \{a\}$.

Now we give a new concept as follows. The circuit graph of a matroid M is a graph $G = G(M)$ with vertex set $V(G)$ and edge set $E(G)$ such that $V(G) = \mathcal{C}$ and $E(G) = \{CC' \mid C, C' \in \mathcal{C}, |C \cap C'| \neq 0\}$, where the same notation is used for the vertices of G and the circuits of M .

Maurer discussed the relationship of bases of matroids and graphs and defined the base graph of a matroid [6, 7]. Liu studied the connectivities of base graph of matroids [3, 4]. Other related results can be found in [1, 5]. In this paper we study the edge connectivity in the circuit graphs of matroids.

In this paper we will prove that the edge connectivity of the circuit graph of any connected matroid is equal to its minimum degree $\delta(G)$.

2 Preliminary Results

Lemma 1 [8]. *A matroid M is connected if and only if for every pair e_1, e_2 of distinct elements of E , there is a circuit containing both e_1 and e_2 .*

Lemma 2 [8]. *If M is a connected matroid, then for every $e \in E$, either M/e or $M \setminus e$ is also connected.*

^{*} This research is supported by NSFC(60673047) and RSDP(20040422004) of China.

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Lemma 3 [8]. Let C and C^* be any circuit and co-circuit of a matroid M . Then $|C \cap C^*| \neq 1$.

Lemma 4 [8]. If M is a matroid and if X and Y are two disjoint sets of elements of M , then $(M \setminus X)/Y = (M/Y) \setminus X$.

3 Main Results

A matroid M is trivial if it has no circuits. In the following all matroids will be nontrivial.

Next we will discuss the properties of the matroid circuit graph. To prove the main results we firstly present the following lemmas which are clearly true.

Lemma 5. Let M be any nontrivial matroid on E and $e \in E$. If G and G_1 are circuit graphs of M and $M \setminus e$, respectively, then G_1 is a subgraph of G induced by $V_1 = V(G) - V_2$ where $V_2 = \{C \mid C \in \mathcal{C}, e \in C\}$.

Obviously the subgraph G_2 of G induced by V_2 is a complete graph. By Lemma 5, G_1 and G_2 are induced subgraphs of G and $V(G_1)$ and $V(G_2)$ partition $V(G)$.

Lemma 6. For any matroid $M = (E, \mathcal{C})$ which has a 2-cocircuit $\{a, b\}$, the circuit graph of M is isomorphic to that of M/a .

Lemma 7. Suppose that $M = (E, \mathcal{C})$ is a connected matroid with two distinct circuits C_1, C_2 such that $|E - (C_1 \cup C_2)| \geq 2$ and for any $e \in E - (C_1 \cup C_2)$, $M \setminus e$ is disconnected. Then M has a co-circuit of cardinality two.

Proof: We shall prove the theorem by induction on $|E - (C_1 \cup C_2)|$. Suppose that $E - (C_1 \cup C_2) = \{e_1 \cup e_2\}$ and $M \setminus e_i$ ($i = 1, 2$) is disconnected. If C_1 and C_2 are in the same component of $M \setminus e_1$, and $\{e_2\}$ is the other component of $M \setminus e_1$, then any circuit containing e_2 contains e_1 . $\{e_1, e_2\}$ is a co-circuit of M because by (C2), if there is a circuit containing e_1 does not contain e_2 then there is a circuit containing e_2 does not contain e_1 , which is a contradiction. If C_1 and C_2 are not in the same component of $M \setminus e_1$ and e_2 is in the same component with C_1 , then $M \setminus e_2$ cannot be disconnected because there is a circuit containing e_1 intersects both C_1 and C_2 but does not contain e_2 . Thus $\{e_2\}$ is another component of $M \setminus e_1$. Therefore $\{e_1, e_2\}$ is a co-circuit of M .

Suppose the result is true for $|E - (C_1 \cup C_2)| = n - 1$. We prove that the result is also true for $|E - (C_1 \cup C_2)| = n > 3$. Let e_1 be an element of $E - (C_1 \cup C_2)$ and $M \setminus e_1$ is disconnected. Hence, by Lemma 2, M/e_1 is connected. Now if e_2 is any element of $E - (C_1 \cup C_2 \cup \{e_1\})$, the matroid $(M \setminus e_2)/e_1$ is disconnected unless $\{e_1\}$ is a component of $M \setminus e_2$. If $\{e_1\}$ is a component of $M \setminus e_2$, then $\{e_1, e_2\}$ is a co-circuit of M . If $(M \setminus e_2)/e_1$ is disconnected for each $e_2 \in E - (C_1 \cup C_2 \cup \{e_1\})$, by Lemma 4, $(M/e_1) \setminus e_2 = (M \setminus e_2)/e_1$, we have $(M/e_1) \setminus e_2$ is disconnected for all $e_2 \in E - (C_1 \cup C_2 \cup \{e_1\})$. By induction, M/e_1 has a co-circuit of cardinality two. A co-circuit of M/e_1 is also a co-circuit of M . We prove the Lemma. $\square$

Now we give the main theorem of this paper.

Theorem 1. Suppose that $G = G(M)$ is the circuit graph of a connected non-trivial matroid $M = (E, \mathcal{C})$ and C_1 and C_2 are distinct vertices of G . Then C_1 and C_2 are connected by $d = \min\{d(C_1), d(C_2)\}$ edge-disjoint paths where $d(C_1)$ and $d(C_2)$ denote the degree of vertices C_1 and C_2 in G , respectively.

Proof: We shall prove the theorem by induction on $|E(M)|$. When $|E(M)| = 3$, each element in M is parallel to another. It is easy to see that $G(M) = K_3$. The theorem is clearly true. Suppose that the result is true for $|E(M)| = n - 1$. We prove that the result is also true for $|E(M)| = n > 3$. Let C_1 and C_2 be any two vertices in G .

There are two cases to distinguish.

Case 1: $(C_1 \cup C_2) = E(M)$. It is easy to see that C_1 and C_2 are both adjacent to any circuit in $\mathcal{C} - \{C_1 \cup C_2\}$ and the conclusion is obviously true.

Case 2: $(C_1 \cup C_2) \neq E(M)$.

There are two subcases to distinguish.

Subcase 2.1: There is an element $e \in E(M) - (C_1 \cup C_2)$ such that $M \setminus e$ is connected. Let $G_1 = G(M \setminus e)$ be the circuit graph of $M \setminus e$ and G_2 be the subgraph of G induced by V_2 where $V_2 = \{C \mid C \in \mathcal{C}, e \in C\}$. Thus C_1 and C_2 are in G_1 . By induction, in G_1 , C_1, C_2 are connected by $d_1 = \min\{d_1(C_1), d_1(C_2)\}$ edge-disjoint paths where $d_1(C_1)$ and $d_1(C_2)$ denote the degree of vertices C_1 and C_2 in G_1 , respectively. Let $\mathcal{P}_1 = \{P_1, P_2, \dots, P_{d_1}\}$ be the family of shortest edge-disjoint paths connecting C_1 and C_2 in G_1 . Without loss of generality, we may assume that $d_1(C_1) \geq d_1(C_2)$. There are two subcases to distinguish.

Subcase 2.1a: $d_1(C_1) = d_1(C_2)$. The conclusion is obvious because G_2 is a complete graph.

Subcase 2.1b: $d_1(C_1) > d_1(C_2)$. By induction, in G_1 there are $d_1 = \min\{d_1(C_1), d_1(C_2)\} = d_1(C_2)$ edge-disjoint paths connecting C_1 and C_2 . Let $\mathcal{P}_1 = \{P_1, P_2, \dots, P_{d_1(C_2)}\}$ be the family of shortest edge-disjoint paths connecting C_1 and C_2 in G_1 . It is obvious that each $P_i (i = 1, 2, \dots, d_1(C_2))$ contains exactly one vertex adjacent to C_1 and one vertex adjacent to C_2 . Let $A_1, A_2, \dots, A_{d_1(C_1) - d_1(C_2)}$ be the vertices in G_1 that are adjacent to C_1 but not contained in d_1 edge-disjoint paths. By Lemma 1, for any element e' in $A_i (i = 1, 2, \dots, d_1(C_1) - d_1(C_2))$ there is a circuit A'_i in G_2 containing e and e' , thus $A_i A'_i$ is an edge in $G(M)$. Let $D_1, D_2, \dots, D_m$ denote the vertices in G_2 that is adjacent to C_2 . G_2 is a complete graph, so A'_i is adjacent to $D_j (i = 1, 2, \dots, d_1(C_1) - d_1(C_2); j = 1, 2, \dots, m)$. If $m \leq d_1(C_1) - d_1(C_2)$, $C_1 A_i A'_i D_i C_2$ are m edge-disjoint paths connecting C_1 and C_2 where A'_i can be $D_i (i = 1, 2, \dots, m)$. Here it is possible that $A'_i = A'_j (i \neq j; i, j = 1, 2, \dots, d_1(C_1) - d_1(C_2))$. But it is forbidden that $D_i = D_j (i \neq j; i, j = 1, 2, \dots, m)$. $d(C_2) = d_1(C_2) + m \leq d_1(C_1) < d(C_1)$, thus $d = \min\{d(C_1), d(C_2)\} = d(C_2)$. $\mathcal{P} = \mathcal{P}_1 \cup \{C_1 A_1 A'_1 D_1 C_2, C_1 A_2 A'_2 D_2 C_2, \dots, C_1 A_m A'_m D_m C_2\}$ are d edge-disjoint paths connecting C_1 and C_2 in G (see Fig.1).

If $m > d_1(C_1) - d_1(C_2)$, The proof is similar to that above.

Fig. 1.

Subcase 2.2: There is no element $e \in E(M) - (C_1 \cup C_2)$ such that $M \setminus e$ is connected. If $E(M) - (C_1 \cup C_2) = \{e\}$ and $M \setminus e$ is disconnected, it is easy to see that $C_1 \cap C_2 = \emptyset$ and C_1, C_2 are the two components of $M \setminus e$. Thus any circuit of M intersecting both C_1 and C_2 contains e , then C_1 and C_2 are both adjacent to any circuit in $\mathcal{C} - \{C_1 \cup C_2\}$ and the conclusion is obviously true. Suppose that $|E(M) - (C_1 \cup C_2)| \geq 2$ and for any $e \in E(M) - (C_1 \cup C_2)$, $M \setminus e$ is disconnected. By Lemma 7, M has a 2-cocircuit $\{a, b\}$. By Lemma 6, the circuit graph of M/a is isomorphic to that of M . By induction hypothesis, the result holds.

Thus the theorem follows by induction. $\square$

By Theorem 1 and Menger's Theorem, we can get the following corollary.

Corollary 2. Suppose that $G = G(M)$ is the circuit graph of a connected matroid M with minimum degree $\delta(G)$. Then the edge connectivity $\kappa'(G) = \delta(G)$.

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