Even Factors with Degree at Most Four in Claw-Free Graphs

Qiuxin Wu¹, Liming Xiong^{2,*}, and Tingzeng Wu³

¹ College of Science Beijing Institute of Machinery
Beijing 100085, P.R. China
qxwu@263.net

² Department of Mathematics, Beijing Institute of Technology
Beijing 100081, P.R. of China
lmxiong@eyou.com

³ Department of Mathematics, Qinghai Nationalities College
Qinghai, 810007, P.R. China

Abstract. In this paper we will prove that every simple claw-free graph G with the minimum degree at least two such that each of its odd branch-bonds contains a branch of length one has an even factor of degree 2 or 4. Some examples showed the conclusion can not be replaced by “2-factor” even for 2-connected graphs.

Keywords: even factor, claw-free graph, branch-bond.

1 Introduction

All graphs considered in this paper are simple graphs. For the notations or terminologies not defined here, see [1].

A *factor* of a graph G is a spanning subgraph. A connected spanning subgraph is called a *connected factor*. A *even factor* of a graph G is a spanning subgraph of G such that it has only even positive degree vertices. A *2-factor* of a graph G is a spanning subgraph such that it has only degree 2 vertices. An even factor of G is called an *even [2, 4]-factor* if the degree of every vertex is either 2 or 4. Clearly, a hamiltonian graph has a 2-factor with exactly one component. There are many results on the existence of the factors with some properties, for the connected factors, see the updated survey papers [5] and [6].

From the results of Choudum and Paulraj [2], and Egawa and Ota [3], one can deduce that every claw-free graph with minimum degree at least 4 has a 2-factor. Very recently, Yoshimoto show that the minimum degree can be reduced if we increase its connectivity.

Theorem 1. (Yoshimoto, [11]) *Every 2-connected claw-free graph G such that $\delta(G) \geq 3$ has a 2-factor.*

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A nontrivial path is called a *branch* if it has only internal vertices of degree two and end vertices of degree not two. The *length of a branch* is the number of its edges. Note that a branch of length one has no internal vertex. A set $\mathcal{B}$ of branches of G is called a *branch cut* if the subgraph of G obtained from $G[E(G) \setminus \bigcup_{B \in \mathcal{B}} E(B)]$ by deleting all internal vertices in any branch of $\mathcal{B}$ has more components than G . A minimal branch cut is called a *branch-bond*. A branch-bond is called *odd* if it has an odd number of branches. The idea of branch-bond is from [9] which has also been applied in [4] and [10].

If G is a graph such that it has an odd branch-bond with a shortest branch of length 2, then clearly G has no even factor. However our result shows that every graph in which every branch-bond has a shortest branch of length one has an even factor.

Theorem 2. *Let G be a claw-free graph with $\delta(G) \geq 2$ such that every odd branch-bond of G contains a branch of length one. Then G has an even $[2, 4]$ -factor.*

Let G_0 be the graph in Figure 1. Then G_0 has no 2-factor since the two vertices of degree 2 in the cycle of length 8 must be in the same cycle of a possible 2-factor but then the vertices of degree 2 having distance 2 must be in a component of maximum degree 4. Note that G_0 satisfies the condition of Theorem 2. This shows that Theorem 2 is best possible in a sense that the conclusion in Theorem 2 can not be replaced by “2-factor”. Note that G_0 is 2-connected. This shows that Theorem 2 is best possible even if the graph is 2-connected.

Fig. 1. The graph G_0

2 Proof of Theorem 2

We start with the following lemma which is a special case of a result in [8].

Lemma 3. *(Li and Xiong [8], Lai, Li, Shao and Xiong [7]) Let G be a claw-free graph. If G has a connected even factor H , then G has a connected even $[2, 4]$ -factor.*

For subgraphs $H \subset F$, let $\text{Int}_F H = \{u \in V(H) \mid d_F(u) \neq 1\}$. A *circuit* is a connected graph with at least three vertices in which every vertex has even degree.

Proof of Theorem 2. Let G be a graph in Theorem 2. Then we have the following fact.

Claim 1. G has a set of vertex-disjoint circuits containing all branches of length not 1.

Proof of Claim 1. Let $C_1, C_2, \dots, C_t$ be vertex-disjoint circuits in G such that $C = \bigcup_{i \leq t} C_i$ contains branches of length not equal to one as many as possible. Let $F = G - V(C)$, and suppose F contains a branch of G with length not one, say P . Hence P has no end-vertex of degree one in G . Since $\text{Int}_G(P) \subset V(F)$, $E(P) \subset E(G) \setminus E(C)$. Let T be a maximal tree such that $P \subset T$ and

if there is an edge e in $T \cap C$, then e is in a branch of G with length one. (2.1)

If we remove all edges and the internal vertices of P from T , then two trees T_1 and T_2 remain. Let $\mathcal{B}$ be a branch-bond of G joining T_1 and $G - V(T_1) \cup \text{Int}_P(P)$ in which P is one of branches.

We choose a branch B of G in $\mathcal{B}$ as follows. If $\mathcal{B} \setminus P$ has a branch of G which is edge-disjoint to C , then let B be the branch. In the case that $\mathcal{B} \setminus P$ has no such a branch, $\mathcal{B}$ is an odd branch-bond, and so $\mathcal{B}$ has a shortest branch of length one. We choose the shortest branch as B . Notice that if $E(B) \cap E(C) \neq \emptyset$, then B is a branch of length one. In either case, because of the maximality of T , B is joining T_1 and T_2 , and so $T \cup B$ contains a cycle D . Then

$$C' = ((C \cup D) \setminus E(C \cap D)) - \text{Int}_{C \cap D}(C \cap D)$$

is a set of circuits. Because $P \subset C'$ and any vertex in $\text{Int}_{C \cap D}(C \cap D)$ is contained in a branch of length one by (2.1), the set C' of the circuits contains more branches of length not equal to one than C , a contradiction. This completes the proof of Claim 1.

By Claim 1, we can choose vertex-disjoint circuits $C_1, C_2, \dots, C_k$ such that

1. $C = \bigcup_{i \leq k} C_i$ contains all edges in any branch of length not equal to one;
2. Subject to 1, $|V(C)|$ is maximal;
3. Subject to the above, k is as small as possible.

We claim that C is a spanning subgraph of G . First we claim that $G - C$ has no cycle for otherwise we will obtain a set of circuits contains bigger k after adding a cycle of $G - C$. Now it suffices to prove that C contains all vertices of G . We will prove this by contradiction. If possible, suppose that $x \in V(G) \setminus V(C)$. By Claim 1, $d_G(x) \geq 3$. Let y_1, y_2 and y_3 be three vertices in $N_G(x)$. Since G is claw-free, there exists an edge between two of $\{y_1, y_2, y_3\}$, say, $y_1 y_2 \in E(G)$.

We obtain a new C' from C by replacing y_1y_2 with y_1xy_2 if $y_1y_2 \in E(C)$ and by adding y_1y_2x otherwise. Note that C' satisfies 1. But C' contains more vertices than C , a contradiction.

Since G is claw-free, $G[C_i]$ is also claw-free. Note that it has an connected even factor C_i . By Lemma 3, $G[C_i]$ has a connected even $[2, 4]$ -factor. Hence G has an even factor with degree at most 4. This completes the proof of Theorem 2.

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