A Linear Algorithm for Edge-Face Coloring Series-Parallel Graphs^{*}

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Abstract. Let G be a series-parallel graph. In this paper, we present a linear algorithm of constructing an **oriented binary decomposition tree** of G . We use it to find 33 unavoidable subgraphs of G . Based on these 33 avoidable subgraphs, we can determine the edge-face chromatic number, denoted by $\chi_{ef}(G)$, of G where G is 2-connected and $\Delta(G) = 5$. This completes the literature of determining $\chi_{ef}(G)$ for 2-connected series-parallel graphs.

Keywords: edge-face coloring, decomposition tree, series-parallel graph.

1 Introduction

In this paper all graphs are simple, undirected and plane. A plane graph G is a particular drawing of a planar graph on the Euclidean plane. Let G be a plane graph with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. An edge-face coloring of a plane graph G is a coloring of $E(G) \cup F(G)$ such that no two adjacent or incident elements receive the same color. A *2-terminal simple series-parallel graph*, denoted by **SP-graph**, is a graph with two distinguished vertices (the terminals) that is obtained as follows:

Base case A single edge uv is a 2-terminal SP-graph with terminals u and v .

Recursive step Let G_1 be a SP-graph with source $u(G_1)$ and terminal $v(G_1)$, and let G_2 be a SP-graph with source $u(G_2)$ and terminal $v(G_2)$.

-A graph G obtained from G_1 and G_2 by identifying $v(G_1)$ and $u(G_2)$ is a SP-graph with source $u(G) = u(G_1)$ and terminal $v(G) = v(G_2)$. Such a connection is called a *series connection*.

-A graph G obtained from G_1 and G_2 by identifying $u(G_1)$ and $u(G_2)$ and $v(G_1)$ and $v(G_2)$ is a SP-graph with source $u(G) = u(G_1) = u(G_2)$ and terminal $v(G) = v(G_1) = v(G_2)$ where $u(G_1)v(G_1)$ and $u(G_2)v(G_2)$ can not be both edges simultaneously in G_1 and G_2 . Such a connection is called a *parallel connection*. It is well-known that the following statements are equivalent if G is a 2-connected graph:

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Figure 1-a

Figure 1-b

Figure 1- c

Fig. 1. Examples of series and parallel connections

- (1) G does not contain K_4 as a minor;
- (2) G is a 2-terminal SP-graph;
- (3) G is a partial 2-tree.

The family of SP-graphs is a distinguished family of plane graphs which is particularly interesting to computer scientists. For example, the problem of determining chromatic index is NP-complete. On the other hand, for plane graphs G , it is not known whether determining $\chi'(G)$ is NP-hard. However, there is a linear algorithm to find $\chi'(G)$ if G is a series-parallel graph [12]. Fast algorithms have been found on the SP-graphs for min-max matching problem, Steiner subgraph and dominating set problems (see [3]). Graph theorists also use SP-graphs in an attempt to solve problems on plane graphs. For example, Seymour's conjecture on chromatic index, if G is a plane graph then $\chi'(G) = \max\{\Delta(G), \lceil \Gamma(G) \rceil\}$, was first proven on outerplane graphs, then on series-parallel graphs (see [5]) where $\Gamma(G) = \max\{\frac{2\bar{X}}{|X|-1} : X \subset V(G), |X| \geq 3 \text{ and odd}\}$ and $\bar{X}$ denotes the set of edges with both ends in X .

In 1975, Mel'nikov [2] conjectured that $\Delta(G) \leq \chi_{ef}(G) \leq \Delta(G) + 3$ for any plane graph G . Waller [7] (independently by Sanders and Zhao [4]) proved the conjecture by using the Four Color Theorem. Wang and Lih [10] recently proved the conjecture without using the Four Color Theorem. Borodin [1] proved that for any plane graph G , $\chi_{ef}(G) \leq \max\{11, \Delta(G) + 1\}$. Wang [8] proved that for any outerplane graph G , $\chi_{ef}(G) \leq \max\{7, \Delta(G) + 1\}$ and $\chi_{ef}(G) = \Delta(G)$ if G is 2-connected and $\Delta(G) \geq 6$. In [9], Wang and Zhang further improved Wang's result and obtained the edge-face chromatic number of outerplane graphs with $\Delta(G) \geq 5$. Wu and Wang [11] extend the previous result to SP-graphs and proved that $\chi_{ef}(G) = \Delta(G)$ if G is a 2-connected SP-graph with $\Delta(G) \geq 6$. This result provides a better upper bound than the one in Borodin's result and encompasses a larger family of plane graphs than the one given by Wang and Zhang in [9]. In the following theorem, we will prove the remaining case where $\Delta(G) = 5$. This is the only case which is still open for these types of theorems in which $\chi_{ef}(G) = \Delta(G)$ with $\Delta(G) \geq c$ because of the following example. Let abc and xyz be two triangles. We form a new graph G by identifying vertices c and x and add a P_2 between a and z . Clearly, $\chi_{ef}(G) = 5$ and $\Delta(G) = 4$. In

fact, a family of graphs constructed similarly with any two odd cycles has the same property. This completes the literature of edge-face coloring of 2-connected SP-graphs.

Theorem 1. *If G is a 2-connected SP-graph, then $\chi_{ef}(G) = 5$ if $\Delta(G) = 5$.*

2 Subgraph Identification

It is well-known that a SP-graph G can be represented by a binary decomposition tree [6]. This technique has been used very successfully to design linear time algorithms to obtain different parameters of SP-graphs (see [12] [13]). In order to apply the binary decomposition tree technique successfully to the face coloring problem, we not only need to construct a binary decomposition tree to represent G but we also require this representation to provide further information about the allocation of faces with respect of its adjacent edges. In the following we shall use the left child and the right child of a node to distinguish whether a face is on the left side or the right side of a particular subgraph. Next we will define an oriented binary decomposition tree of G , denoted by T_O , according to the binary decomposition tree of G , denoted by T , where T is constructed by the algorithm provided by Takamizawa et.al. in [6].

Lemma 1. *Let G be a 2-terminal SP-graph with terminals u and v . Then there exists an acyclic edge orientation of G such that u is the only source and v is the only terminal.*

This can be easily showed by using induction to the connection steps in the definition of a 2-terminal simple SP-graph. By Lemma 1, we can assign an acyclic edge orientation to G from the source to the sink of G . Let G_1 be a SP-graph with source $u(G_1)$ and terminal $v(G_1)$, and let G_2 be a SP-graph with source $u(G_2)$ and terminal $v(G_2)$. For any S node in the binary decomposition tree of T , we assign G_1 as its left child and G_2 as its right child in T_O if $v(G_1) = u(G_2)$. For example, Figure 1-a shows a binary decomposition tree which represents a series connection of two oriented edges. For any P node in the binary decomposition tree of T , we assign G_1 as its left child and G_2 as its right child in T_O if G_1 is on the left side of G_2 when facing from the source to the terminal in G_2 . Otherwise we assign G_2 as its the left child and G_1 as its right child in T_O if G_1 is on the right side of G_2 when facing from the source to the terminal in G_2 . For example, Figure 1-b and Figure 1-c show two binary decomposition trees which represent two possible parallel connections for the case where $G_1 = uv$ and $G_2 = uvv$.

Clearly, all internal nodes are either S or P nodes and all leaves are edges of G . We shall use a sequence SXY (PXY) to denote a subtree of T_O where X and Y are the left child and the right child of S (P) which can be either a subtree or a leaf ($e \in E(G)$). Since we have to color faces of G , PXY and PYX might be different. By the symmetry of subgraphs, most of SXY and SYX are the same.

Theorem 2. *Let G be a 2-connected SP-graph. Then G must contain at least one of the following 33 unavoidable subgraphs (see Figure 2).*

Fig. 2. 33 unavoidable subgraphs

Proof. Let $P\{t_k, t_{k-1}, \dots, t_1, t_0\}$ be the longest path in T_O where $t_0 = e$ is a leaf node and t_k is the root node. Since G is a simple graph, the last internal node t_1 must be a S node. For convenience, let $A_0 = See$ (see Figure 3). In the following, we shall move up on the longest path P and find all the desired subgraphs. Since the nodes t_0 and t_1 have been discussed above, we shall start from t_i where $i \geq 2$ in the following cases.

Case 1. $i = 2$

If $t_2 = S$, then one of its two children of t_2 must be A_0 and the other can be either A_0 or a leaf node. In all three cases, s_1 is found. If $t_2 = P$, then there are three possible combinations and we call the three resulting subgraphs $A_1 = PA_0e$, $A'_1 = PeA_0$ and $A_2 = PA_0A_0$ (see Figure 3).

Fig. 3. A_i for $i = 1, 2, \dots, 6$

Case 2. $i = 3$

One of the children of t_3 must be in $\{A_1, A'_1, A_2\}$.

Subcase 2.1. $t_3 = S$

SA_1A_2 and SA'_1A_2 leads $s_4 \subseteq G$. SA_1e leads to $s_2 \subseteq G$. SA_2e leads $s_3 \subseteq G$. Let $A_3 = SA_1A_1$, $A'_3 = A'_1A'_1$, $A''_3 = A_1A'_1$ and $A'''_3 = A'_1A_1$ (see Figure 3).

Subcase 2.2. $t_3 = P$

It is impossible to have PXY where X and $Y \subseteq \{e, A_1, A'_1\}$ because G is a simple graph. PeA_2 , PA_2e , PA_1A_2 , PA_2A_1 , PA'_1A_2 , $PA_2A'_1$, PA_0A_1 and PA'_1A_0 lead to $s_9 \subseteq G$. PA_2A_2 leads to $s_7 \subseteq G$. Let $A_4 = PA_0A'_1 = PA_1A_0$ and $A_5 = PA_0A_2 = PA_2A_0$ (see Figure 3).

Case 3. $i = 4$

One of the children of t_4 must be in $\{A_3, A'_3, A''_3, A'''_3, A_4, A_5\}$. Since $\Delta(G) = 5$, we will not consider the combinations of A_4 and A_4 , A_4 and A_5 , and, A_5 and A_5 .

Subcase 3.1. $t_4 = S$

SXY leads to $s_2 \subseteq G$ if $X \in \{e, A_0\}$ and $Y \in \{A_3, A'_3, A''_3, A'''_3\}$. SXY implies that one of $\{s_6, s'_6, s''_6\}$ is a subgraph of G if $X \in \{A_1, A'_1, A_3, A'_3, A''_3, A'''_3\}$ and $Y \in \{A_3, A'_3, A''_3, A'''_3\}$. SA_2Y leads to $s_4 \subseteq G$ if $Y \in \{A_3, A'_3, A''_3, A'''_3\}$. SA_4Y leads to $s_{13} \subseteq G$ if $Y \in \{A_3, A'_3\}$. SA_5Y leads to $s_{10} \subseteq G$ if $Y \in \{A_3, A'_3\}$. SA_4Y leads to $s_{12} \subseteq G$ if $Y \in \{e, A_0\}$, to $s_{13} \subseteq G$ if $Y \in \{A_1, A'_1\}$, and to $s_{14} \subseteq G$ if $Y = A_2$. SA_5Y leads to $s_8 \subseteq G$ if $Y \in \{e, A_0\}$, to $s_{10} \subseteq G$ if $Y \in \{A_1, A'_1\}$, and to $s_{11} \subseteq G$ if $Y = A_2$.

Subcase 3.2. $t_4 = P$

Let $A_6 = PA_3e = PeA'_3$, $A'_6 = PA'_3e = PeA_3$, $A''_6 = PA''e = PeA'''_3$ and $A'''_6 = PA'''_3e = PeA''_3$ (see Figure 3). PA_3A_0 and PA'_3A_0 lead to $s_{25} \subseteq G$. PA_0A_3 and PA'_3A_0 lead to $s_{24} \subseteq G$. PXY and PYX lead to $s'_{24} \subseteq G$ if $X = A_0$ and $Y \in \{A'_3, A''_3\}$. PA_3A_1 and $PA'_1A'_3$ lead to $s_{20} \subseteq G$, PA_1A_3 and $PA'_3A'_1$ lead to $s_{19} \subseteq G$, $PA_3A'_1$ and $PA_1A'_3$ lead to $s_{18} \subseteq G$, and PA'_3A_1 and PA'_1A_3 lead to $s_{21} \subseteq G$. PXY and PYX lead to either $s'_{18} \subseteq G$ or $s'_{20} \subseteq G$ if $X \in \{A_1, A'_1\}$ and $Y \in \{A'_3, A''_3\}$. PXA_2 and PA_2X lead to $s_4 \subseteq G$ if $X \in \{A_3, A'_3, A''_3, A'''_3\}$. PA_3A_3 and $PA'_3A'_3$ lead to $s_{16} \subseteq G$. $PA_3A'_3$ leads to $s_{15} \subseteq G$. And PA'_3A_3 leads to $s_{17} \subseteq G$. $PA''_3A''_3$ and $PA'''_3A'''_3$ lead to $s'_{15} \subseteq G$. $PA''_3A'_3$ and $PA'''_3A''_3$ lead to $s''_{15} \subseteq G$. PXA_4 and PA_4X lead to either $s_{20} \subseteq G$ or $s_{21} \subseteq G$ if $X \in \{A_3, A'_3\}$. PXA_4 and PA_4X lead to $s'_{20} \subseteq G$ if $X \in \{A_3, A'_3\}$. PXA_5 and PA_5X lead to $s_{10} \subseteq G$ if $X \in \{A'_3, A''_3\}$. It is impossible to have PXA_4 and PA_4X where $X \in \{e, A_1\}$ because G is a simple graph. PA_4A_2 and PA_2A_4 lead to $s_9 \subseteq G$. PA_5e and PeA_5 lead to $s_9 \subseteq G$. PA_5X and PXA_5 lead to $s_7 \subseteq G$ if $X \in \{A_0, A_2\}$. PA_5A_1 and PA'_1A_5 lead to $s_7 \subseteq G$. $PA_5A'_1$ and PA_1A_5 lead to $s_9 \subseteq G$.

Case 4. $i = 5$

One of the children of t_5 must be in $\{A_6, A'_6, A''_6, A'''_6\}$. Since $\Delta(G) = 5$, we will not consider the subtrees SXY , PXY and PYX where $X \in \{A_6, A'_6, A''_6, A'''_6\}$ and $Y \in \{A_4, A_5, A_6, A'_6, A''_6, A'''_6\}$.

Subcase 3.1. $t_5 = S$

SA_6Y leads to $s_{26} \subseteq G$ if $Y = \{e, A_0\}$, to either $s_{28} \subseteq G$ or $s_{30} \subseteq G$ if $Y \in \{A_1, A'_1, A_3, A'_3\}$, and to $s_{32} \subseteq G$ if $Y = A_2$. SA'_6Y leads to $s_{27} \subseteq G$ if $Y \in \{e, A_0\}$, to either $s_{29} \subseteq G$ or $s_{31} \subseteq G$ if $Y \in \{A_1, A'_1, A_3, A'_3\}$, and to $s_{33} \subseteq G$ if $Y = A_2$. SA''_6Y and SA'''_6Y lead to either $s'_{26} \subseteq G$ or $s''_{26} \subseteq G$ if $Y \in \{e, A_0\}$, to either $s'_{28} \subseteq G$ or $s''_{28} \subseteq G$ if $Y \in \{A_1, A_3\}$, to either $s'_{30} \subseteq G$ or $s''_{30} \subseteq G$ if $Y \in \{A'_1, A'_3\}$, and to either $s'_{32} \subseteq G$ or $s''_{32} \subseteq G$ if $Y = A_2$.

Subcase 3.1. $t_5 = P$

Since G is a simple graph, we will not consider the subtrees PXY and PYX where $X \in \{A_6, A'_6, A''_6, A'''_6\}$ and $Y \in \{e, A_1, A'_1\}$.

PA_6A_0 and PA_6A_2 lead to $s_{18} \subseteq G$. PXY and PYX lead to either $s_{28} \subseteq G$ or $s_{30} \subseteq G$ if $X = A_6$ and $Y \in \{A_3, A'_3, A''_3, A'''_3\}$. PA_0A_6 and PA_2A_6 lead to $s_{22} \subseteq G$. PA'_6A_0 and PA'_6A_2 lead to $s_{19} \subseteq G$. PXY and PYX lead to either $s_{29} \subseteq G$ or $s_{31} \subseteq G$ if $X = A'_6$ and $Y \in \{A_3, A'_3, A''_3, A'''_3\}$. $PA_0A'_6$ and $PA_2A'_6$ lead to

$s_{23} \subseteq G$. PXY leads to $s'_{18} \subseteq G$ and PYX leads $s'_{23} \subseteq G$ if $X \in \{A''_6, A'''_6\}$ and $Y \in \{A_0, A_2\}$. PXY and PYX lead to either $s'_{30} \subseteq G$ or $s''_{30} \subseteq G$ if $X \in \{A''_6, A'''_6\}$ and $Y = A_3$, and to either $s'_{28} \subseteq G$ or $s''_{28} \subseteq G$ if $X \in \{A''_6, A'''_6\}$ and $Y = A'_3$. This completes the proof.

Since the algorithm we used to construct T given by Takamizawa et.al. in [6] is a linear algorithm in terms of $|E(G)|$ and we traverse T once to construct T_O . We may traverse the whole tree T_O once in the worst case to find one of the 33 unavoidable subgraphs in G . Therefore, one can find one of the 33 unavoidable subgraphs in G in $O(|E(G)|)$. This technique of constructing an oriented binary decomposition tree from any SP-graph can be applied to future research on the problems that are related to the faces and edges of any SP-graph. As an example, we will use it to prove Theorem 1 in the next section.

3 Proof of Theorem 1

Proof. We shall prove the theorem by induction on $|V(G)| + |E(G)|$. Clearly, the result is obvious if $|V(G)| \leq 4$. Let $G' \subset G$ and ϕ be an edge-face coloring of G' where $\phi : (EF(G)) \rightarrow C = \{1, 2, \dots, 5\}$. We shall extend ϕ to an edge-face coloring σ of G with t colors. For convenience, we denote $T(v) = C \setminus \{\phi(vv') : v' \in N(v)\}$, the set of colors missing at vertex v . In Figure 2, let f denote the face on the left side of uv and f' denote the face on the right side of uv when facing from u to v , and, without loss of generality, let $\phi(f) = 1$ and $\phi(f') = 2$.

For s_1 , let G' be the result graph by identifying vertices x_1 and x_2 in G . Then ϕ can be easily extend to G since $d(x_1) = d(x_2) = 2$ and $|C| \geq 5$. For s_2 , let $G' = G \setminus \{x_1\}$ and $c_1 \in T(u)$. If $c_1 = 1$, then we assign $\sigma(ux_2) = 1, \sigma(ux_1) = 2, \sigma(vx_2) = \sigma(ux_1x_2) = 4$, and $\sigma(x_1x_2) = 5$. If $c_1 \neq 1$, we can assign $\sigma(ux_2) = 3, \sigma(vx_2) = 4, \sigma(ux_1) = c_1; \sigma(x_1x_2) = 2$ and $\sigma(ux_1x_2) = \{4, 5\} - c_1$ if $c_1 \neq 2$ or $\sigma(x_1x_2) = 5$ and $\sigma(ux_1x_2) = 4$ if $c_1 = 2$.

For s_3 , let $G' = G \setminus \{x_1, x_2\} + ux_3$ and $c_1 \in T(u)$. If $c_1 = 1$, then we assign $\sigma(ux_2) = 1, \sigma(ux_1) = \sigma(x_2x_3) = 3, \sigma(vx_3) = \sigma(ux_1x_3x_2) = 4$ and $\sigma(x_1x_3) = 5$. If $c_1 \neq 1$, we can assign $\sigma(ux_1) = c_1, \sigma(x_2x_3) = 1, \sigma(ux_2) = \sigma(x_1x_3) = 3, \sigma(vx_3) = 4$ and $\sigma(ux_1x_3x_2) = \{4, 5\} - c_1$.

For s_4 , let $G' = G \setminus \{x_1, x_2, x_4\} + ux_3$. Without lose of generality, we may assume that $\phi(ux_3) = 3$ and $\phi(x_3v) = 4$. Let $c_1 \in T(u)$ and $c_2 \in T(v)$. If $c_1 = c_2 = 1$, then we can assign $\sigma(ux_2) = \sigma(vx_3) = 1, \sigma(x_3x_4) = 2, \sigma(ux_1) = \sigma(x_2x_3) = 3, \sigma(vx_4) = \sigma(ux_1x_3x_2) = 4$, and $\sigma(vx_3x_4) = \sigma(x_1x_3) = 5$. Now consider the case where $c_1 = 1$ and $c_2 \neq 1$. We can assign $\sigma(ux_2) = 1, \sigma(ux_1) = \sigma(x_2x_3) = 3, \sigma(vx_3) = \sigma(ux_1x_3x_2) = 4, \sigma(vx_4) = c_2; \sigma(x_1x_3) = 2, \sigma(x_3x_4) = 5$ and $\sigma(vx_3x_4) = 3$ if $c_2 = 2$ or $\sigma(x_1x_3) = 5, \sigma(x_3x_4) = 2$ and $\sigma(vx_3x_4) = \{3, 5\} - c_2$. Note that the case where $c_1 \neq 1$ and $c_2 = 1$ can be solved similarly. Finally, we consider the case where $c_1 \neq 1$ and $c_2 \neq 1$. We assign $\sigma(ux_1) = c_1, \sigma(x_4v) = c_2, \sigma(ux_2) = \sigma(x_3x_1) = 3, \sigma(vx_3) = 4, \sigma(x_2x_3) = 1$, and $\sigma(ux_1x_3x_2) \in \{4, 5\} \setminus c_1; \sigma(x_3x_4) = 5, \sigma(vx_3x_4) = 3$ if $c_2 = 2$ or $\sigma(x_3x_4) = 2, \sigma(vx_3x_4) \in \{3, 5\} \setminus c_2$ if $c_2 \neq 2$.

The rest of proof is similar with these cases. We shall leave it on the following website www.stfx.ca/people/pwang/restproof.pdf.

Remarks. The process of coloring these 33 unavoidable subgraphs takes a constant time. Hence, one can color the edges and faces of G with 5 colors in $O(|E(G)|)$ time.

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