On (s, t) -Supereulerian Locally Connected Graphs*

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Abstract. For two integers $s \geq 0, t \geq 0$, a graph G is (s, t) -supereulerian, if $\forall X, Y \subset E(G)$, with $X \cap Y = \emptyset, |X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H with $X \subset E(H)$ and $Y \cap E(H) = \emptyset$. We prove that if G is $(t + 2)$ -edge-connected and locally connected, then G is $(2, t)$ -supereulerian, or G belongs to a well characterized class of graphs.

Keywords: (s, t) -supereulerian, Collapsible graph, Locally connected graphs, Reduction.

1 Introduction

Graphs in this note are simple and finite. We follow the notation of Bondy and Murty [2] unless otherwise stated. For a graph G , $O(G)$ denotes the set of all vertices of odd degree in G . A graph G with $O(G) = \emptyset$ is an *even graph*, and a connected even graph is an *eulerian graph*. A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by SL . For a graph G with a subgraph H , the contraction G/H is the graph obtained from G by replacing H by a vertex v_H , such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H . v_H is *nontrivial* if $E(H) \neq \emptyset$. As in [2], the edge-connectivity and the minimum degree of G are denoted by $\kappa'(G)$ and $\delta(G)$. For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) | d_G(v) = i\}$. A vertex $v \in G$ is called a *locally connected* vertex if $G[N_G(v)]$ is connected. A graph is a *locally connected* if every $v \in G$ is a locally connected.

For two integers $s \geq 0, t \geq 0$, G is (s, t) -supereulerian, if $\forall X, Y \subset E(G)$, with $X \cap Y = \emptyset, |X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H with $X \subset E(H)$ and $Y \cap E(H) = \emptyset$. Clearly, G is supereulerian if and only if G is $(0, 0)$ -supereulerian. Since every supereulerian graph must be 2-edge-connected, it is necessary that any (s, t) -supereulerian graph must be $(t + 2)$ -edge-connected. We consider the problem of $s = 2$ and shall prove that if G is $(t + 2)$ -edge-connected and locally connected, then either G is $(2, t)$ -supereulerian or G belongs to a well characterized class of graphs. In [3], Catlin proved the following.

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Theorem 1 (Catlin [3]). If G is connected and locally connected, then $G \in SL$.

The main purpose of this paper is to improve this theorem of Catlin on $(0, 0)$ -supereulerian graphs to $(2, t)$ -supereulerian graphs.

2 Collapsible Graphs and Reduced Graphs

A graph is *collapsible* if for every set $R \subset V(G)$ with $|R|$ even, there is a spanning connected subgraph H_R of G , such that $O(H_R) = R$. Thus K_1 is both supereulerian and collapsible. Denote the family of collapsible graphs by CL . Let G be a collapsible graph and let $R = \emptyset$. Then by definition, G has a spanning connected subgraph H with $O(H) = \emptyset$, and so G is supereulerian. Therefore, we have $CL \subset SL$.

In [3], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \dots, H_c$. The contraction of G obtained from G by contracting each H_i into a single vertex ($1 \leq i \leq c$), is called the *reduction* of G . A graph is *reduced* if it is the reduction of some other graph.

Theorem 2 (Lai [9]). Let G be a 2-edge-connected triangle-free simple graph on $n > 30$ vertices, if $\delta(G) > \frac{n}{10}$, then $G \in SL$.

Theorem 3 (Catlin, Theorem 8 of [3]). Let H be a collapsible subgraph of a graph G , then $G \in SL$ if and only if $G/H \in SL$.

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2-edge-disjoint spanning trees.

Theorem 4 (Catlin, Han and Lai, Lemma 2.3 of [6]). If G is reduced with $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

Theorem 5 (Catlin, Theorem 7 of [4], Catlin, Han and Lai, Theorem 1.3 of [6]). Let G be a connected reduced nontrivial graph. If $F(G) \leq 1$, then $G \cong K_2$; if $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t} \ (t \geq 1)\}$.

Theorem 6 (Catlin, Theorem 8 and Lemma 5 of [3]). If G is reduced, then G is simple and contains no K_3 . Moreover, if $\kappa'(G) \geq 2$, then $\sum_{i=2}^4 |D_i(G)| \geq 4$, and when $\sum_{i=2}^4 |D_i(G)| = 4$, G must be eulerian.

3 Main Result

Let G be a graph and let $X \subset E(G)$. Define G_X to be the graph obtained from G by replacing each edge $e \in X$ with end u_e and v_e by a (u_e, v_e) -path P_e of length 2, where the internal vertex $w(e)$ of the path P_e is newly added. We call this process of replacing e by P_e an elementary subdivision of G on the edge e . Lemma 7 below follows from the definitions.

Lemma 7. Let G be a graph and let $X \subset E(G)$, $Y \subset E(G)$ with $X \cap Y = \emptyset$, then G has a spanning eulerian graph H such that $X \subset E(H)$ and $Y \cap E(H) = \emptyset$ if and only if $(G - Y)_X \in SL$.

Let G_0 be the reduction of $(G - Y)_X$. Let D_2^1 denote the set of those vertices of degree 2 newly created from subdividing edges of X , and let $D_2^2 = D_2(G_0) - D_2^1$. Let $d_2^1 = |D_2^1|$, $d_2^2 = |D_2(G_0)| - d_2^1$.

Lemma 8. Let G be $(t + 2)$ -edge-connected, then $d_2^2 \leq 2$.

Proof: Let $v \in D_2^2$ and the two edges incident with v in G_0 is an edge-cut in $(G - Y)_X$. Since G is $(t + 2)$ -edge-connected, so at least t edges incident with v are removed. Since in order to get three vertices that belong to D_2^2 , at least $\frac{3}{2}t$ edges must be removed, by $|Y| \leq t$, $d_2^2 \leq 2$.

Lemma 9. (Catlin, Lemma 1 of [7]) $K_{3,3} - e$ is collapsible.

Theorem 10. Let $s \leq 2$ and $t \geq 0$ be two integers. G is a $(t + 2)$ -edge-connected locally connected graph on n vertices. For any $X, Y \subset E(G)$ with $X \cap Y = \emptyset$, $|X| \leq s$, $|Y| \leq t$, exactly one of the following holds:

- (i) G has a spanning eulerian graph H , such that $X \subset E(H)$ and $Y \cap E(H) = \emptyset$. (e.g. G is $(2, t)$ -supereulerian).
- (ii) The reduction of $(G - Y)_X$ is a member of $\{K_1, K_2, K_{2,t} (t \geq 1)\}$.

Proof: Let G_0 be the reduction of $(G - Y)_X$. We consider these cases.

Case 1. $F(G_0) \geq 3$.

If $|V(G_0)| \geq 3$, then by Theorem 4, we have

$$F(G_0) = 2|V(G_0)| - |E(G_0)| - 2 \geq 3. \quad (1)$$

Let $d_j = |D_j(G_0)|$, $j = 1, 2, \dots$. Since G is $(t + 2)$ -edge-connected, G_0 must be 2-edge-connected, and so

$$|V(G_0)| = \sum_{j \geq 2} d_j, \quad 2|E(G_0)| = \sum_{j \geq 2} j d_j.$$

By (1)

$$4 \sum_{j \geq 2} d_j - \sum_{j \geq 2} j d_j - 4 \geq 6.$$

By algebraic manipulation, we have,

$$2d_2 + d_3 - \sum_{j \geq 5} (j - 4)d_j - 10 \geq 0,$$

and so

$$2d_2^1 + 2d_2^2 + d_3 - \sum_{j \geq 5} (j - 4)d_j - 10 \geq 0.$$

As $s \leq 2$, we have

$$4 + 2d_2^2 + d_3 - \sum_{j \geq 5} (j-4)d_j - 10 = 2d_2^2 + d_3 - \sum_{j \geq 5} (j-4)d_j - 6 \geq 0. \quad (2)$$

For any $v \in D_2^2 \cup D_3$, by Lemma 8, we have $d_2^2 \leq 2$. By (2), $(d_2^2, d_3) \notin \{(2, 1), (1, 1), (1, 2), (1, 3)\}$. In the following we will proceed our proof by considering the possible values of t .

Subcase 1.1. $t \geq 3$.

Then, $|Y| = t \geq 3$. Take $v \in D_3$, as shown in the Lemma 8, the three edges incident with v is an edge-cut of $(G - Y)_X$. Since G is $(t+2)$ -edge-connected, at least $t-1$ edges incident with v are removed. Since in order to get three vertices in D_3 , at least $\frac{3}{2}(t-1)$ edges must be removed, by $|Y| \leq t$ and $\frac{3}{2}(t-1) \geq t$ while $t \geq 3$, $d_3 \leq 3$.

When $d_3 = 3$, at least $\frac{3}{2}(t-1)$ edges are removed. As $\frac{3}{2}(t-1) \geq t$ (the equal sign holds if and only if $t = 3$), we have $d_2^2 = 0$. By (2), it is a contradiction.

Similarly, $d_3 = 2$ implies $d_2^2 \leq 1$. It contradicts to (2).

If $d_3 = 1$, by Lemma 8, $d_2^2 \leq 2$, (2) can't hold.

So, when $F(G_0) \geq 3$ and $\kappa'(G) \geq 5$, $|V(G_0)| \leq 2$. If $|V(G_0)| = 2$, G_0 is simple, and so we must have $G_0 \cong K_2$. So at least $t+1$ edges are removed. It contradicts to $|Y| \leq t$. Thus $|V(G_0)| = 1$, $G_0 \in SL$.

Subcase 1.2. $t = 2$.

Similarly, if $D_3 \geq 5$, then at least $\frac{5}{2}(t-1)$ edges must be removed, and so $2 = t \geq |Y| \geq \frac{5}{2}(t-1)$, contrary to the fact that $t = 2$. This shows that $d_3 \leq 4$.

If $d_3 = 2$, by (2), $d_2^2 \geq 2$. But by Lemma 8, when $d_2^2 = 2$, we must remove t edges in Y . This contradicts to $d_3 = 2$. So, $d_3 = 2$ is impossible.

Similarly, if $d_3 = 3$, by (2), $d_2^2 \geq 2$. As the discussion above, that contradicts to $d_3 = 3$. Hence $d_3 = 3$ is impossible.

Note that when $d_3 = 4$, $d_2^2 = 0$. And by (2), $d_2^2 \geq 1$, another contradiction. This shows that $d_3 = 4$ is impossible.

As the shown in the Subcase 1.1, $|V(G_0)| = 1$, then $G_0 \in SL$.

Subcase 1.3. $t = 1$.

Since G is locally connected, and $\kappa'(G) \geq 3$, so any v in G , there is a path connecting its neighbouring vertices. Thus any v is in a triangle. By Theorem 6, G_0 has no K_3 . So a locally connected graph will be reduced to K_1 . Hence any $v \in D_3$, any e in $(G[N(v)])$ is either divided or removed. Thus when $s = 2$, $t = 1$, $d_3 = 2$, G_0 must be depicted as the graph in Figure 1.

Clearly, this graph is $(2, 1)$ -supereulerian.

When $d_3 = 3$, since G is locally connected, G_0 must have $K_{3,3} - e$. This contradicts to Lemma 9. Thus $d_3 = 3$ is impossible.

When $d_3 = 4$, by Lemma 9, G_0 has no $K_{3,3}$ -e. So G_0 must be depicted as the graph in Figure 2. But $v_3 \in D_3$, the edge in $(G[N(v_3)])$ incident with a and

Fig. 1.

Fig. 2.

b can't be divided or removed. So, we get a contradiction. So $G_0 \in CL$, thus $G \in SL$.

If $d_3 > 4$, $K_{3,3} - e$ will appear. So, $d_3 \leq 4$.

Case 2. $F(G_0) \leq 2$.

By Theorem 5, $G_0 \in \{K_1, K_2, K_{2,t} (t \geq 1)\}$. This proves Theorem 10.

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