A Degree-Condition for (s, t) -Supereulerian Graphs^{*}

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Abstract. For two integers $s \geq 0, t \geq 0$, G is (s, t) -supererlian, if $\forall X, Y \subset E(G)$ with $X \cap Y = \phi$, where $|X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H such that $X \subset E(H)$ and $Y \cap E(H) = \phi$. It is obvious that G is supereulerian if and only if G is $(0, 0)$ -supereulerian. In this note, we have proved that when G is a $(t + 2)$ -edge-connected simple graph on n vertices, if $n \geq 21$ and $\delta(G) \geq \frac{n}{5} + t$, then G is $(3, t)$ -supererlian or can be contracted to some well classified special graphs.

Keywords: (s, t) -supereulerian; Collapsible; Reduced Graph; Edge-disjoint Spanning Trees.

1 Introduction

Graphs in this note are simple, finite and loopless. Undefined terms and notations are from [2]. We use $H \subseteq G$ to denote the fact that H is a subgraph of G . For a graph G , $O(G)$ denotes the set of all vertices of odd degree in G . A graph G is called an *even* graph if $O(G) = \phi$, and a connected even graph is an *eulerian* graph. A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by SL . For the literature on the subject of supereulerian graphs, see Catlin's excellent survey [4]. As indicated by the authors in [1], characterizing supereulerian graphs appears very difficult. Pulleyblank in [8] pointed out that the problem of determining if a graph G is supereulerian is $NP - complete$.

In [7] Jaeger proved Theorem 1.1 below.

Theorem 1.1(Jaeger [7]). If G has two edge-disjoint spanning trees, then G is in SL .

Let $F(G)$ be the minimum number of edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. Thus Theorem 1.1 says that if $F(G) = 0$, then $G \in SL$. In [5], Catlin showed that Theorem 1.1 can be improved.

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Theorem 1.2(Catlin [5]). If $F(G) \leq 2$, then either G is in SL , or G can be contracted to K_2 or $K_{2,t}$, where t is odd natural number.

In [3], Catlin got a degree condition for supererlerian graphs.

Theorem 1.3(Catlin [3]). Let G be a 2-edge-connected simple graph on n vertices. If $\delta(G) \geq \frac{n}{5}$, then G is supereulerian.

In this note, we want to improve the Theorem 1.3. We need the following concept. For two integers $s \geq 0, t \geq 0$, G is (s, t) -superlerian, if $\forall X, Y \subset E(G)$ with $X \cap Y = \phi$, where $|X| \leq s, |Y| \leq t$, G has a spanning eulerian subgraph H such that $X \subset E(H)$ and $Y \cap E(H) = \phi$. It is obvious that G is supereulerian if and only if G is $(0, 0)$ -supereulerian. The Theorem 1.3 then says G is $(0, 0)$ -supereulerian if $\delta(G) \geq \frac{n}{5}$ with edge-connectivity ≥ 2 . In section 2, we shall display some preliminaries and in section 3, we shall extend the Theorem 1.3 to its (s, t) -supererlerian version.

2 Preliminaries

In [3], Catlin defined the *collapsible* graphs. Let $R \subseteq V(G)$, a graph H of G is called an R -subgraph if both $G - E(H)$ is connected and $v \in R$ if v has odd degree in H . A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R -subgraph. Catlin showed in [4] that every vertex of G is lying in a unique maximal collapsible subgraph of G . The collection of all collapsible graphs is denoted by CL . Clearly $CL \subseteq SL$.

The contraction G/H is obtained from G by contracting each edge of H and deleting the resulting loops. The *reduction* of G is obtained from G by contracting all maximal collapsible subgraphs, and is denoted by G' . A graph G is reduced if G is the reduction of some graph.

Theorem 2.1(Catlin [3]). Let G be a graph. Each of the following holds:

- (1) G is reduced if and only if G has no nontrivial collapsible subgraphs;
- (2) If G is reduced, then G is simple and contains no K_3 , $\delta(G) \leq 3$;
- (3) If G is reduced and $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2 \geq 2$;
- (4) If H is a collapsible subgraph of G , then $G \in CL$ if and only if $G/H \in CL$;
- (5) If H is a collapsible subgraph of G , then $G \in SL$ if and only if $G/H \in SL$.

Let G be a graph and let $X \subseteq E(G)$. The graph G_X is obtained from G by replacing each edge $e \in X$ with ends u_e and v_e by a (u_e, v_e) -path P_e of length 2, where the internal vertex w_e of the path P_e is newly added.

Lemma 2.2. Let G be a graph and let $X \subset E(G), Y \subset E(G)$ with $X \cap Y = \phi$, then G has a spanning eulerian subgraph H such that $X \subset E(H)$ and $Y \cap E(H) = \phi$.

Lemma follows from the definitions.

We need the following definition. The Peterson graph is denoted by P .

Let $\mathcal{F} = \{K_1, K_2, K_{2,t}, K_{2,t}', K_{2,t}'', K_{1,3}(s, s', s''), S(m, l), J(m, l), J'(m, l), P\}$, where t, s, s', s'', m, l are nonnegative integers. The definitions of $K_{2,t}', K_{2,t}'', K_{1,3}(s, s', s''), S(m, l), J(m, l), J'(m, l)$, please see the reference [6].

Theorem 2.3(Chen and Lai [6]). If G is a connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.

Let $\mathcal{F}^* = \{G \in \mathcal{F} : G \text{ is 2-edge-connected}\} \cup \{G : G \text{ is the subdivision of } K_{2,t}, t \text{ is odd natural number}\} \cup F_1 - SL$, here F_1 denotes the graph obtained from a $K_{2,3}$ and C_4 by identifying a vertex of degree 2 in $K_{2,3}$ with any one in C_4 .

3 Main Result

Theorem 3.1. G is a $(t+2)$ -edge-connected simple graph on n vertices, where t is a nonnegative integer. If $n \geq 21$ and $\delta(G) \geq \frac{n}{5} + t$, take any two edge-disjoint subsets of $E(G)$, X and Y , where $|X| \leq 3$ and $|Y| \leq t$, then G is $(3, t)$ -supereulerian or $(G - Y)_X'$ is in $\mathcal{F}^*$.

Note that in Theorem 3.1, G need to be $(2+t)$ -edge-connected because that $G - Y$ is supereulerian only when $G - Y$ is 2-edge-connected. Before the proof of the theorem 3.1, we need the following lemma:

Lemma 3.2. G is a connected simple graph on n vertices. G' is the reduction of G . $V_5 = \{v \in V(G') : d_{G'}(v) < 5\}$. If $\delta(G) \geq 5$, then $|V_5| \leq 4$.

Proofs of Theorem 3.1 and Lemma 3.2 are omitted because of the length.

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