Hamiltonian Connected Line Graphs^{*}

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Abstract. Thomassen conjectured [8] that every 4-connected line graph is hamiltonian. An hourglass is a graph isomorphic to $K_5 - E(C)$, where C is a cycle of length 4 in K_5 . In [2], it is shown that every 4-connected line graph without an induced subgraph isomorphic to the hourglass is hamiltonian connected. In this note, we prove that every 3-connected, essentially 4-connected hourglass-free line graph is hamiltonian connected.

Keywords: Line Graph, Hourglass, Hourglass-Free, Hamiltonian Connected.

1 Introduction

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [1] for notations and terms. A graph G is **nontrivial** if $E(G) \neq \emptyset$. For a vertex v of a graph G , $d_G(v)$ denotes the degree of v in G and $E_G(v)$ denotes the set of edges incident with v in G . For an integer $i > 0$, $D_i(G) = \{v \in V(G) : d_G(v) = i\}$.

The **line graph** of a graph G , denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Thomassen [8] conjectured that every 4-connected line graph is hamiltonian. This conjecture is still open.

An hourglass is a graph isomorphic to $K_5 - E(C)$, where C is a cycle of length 4 in K_5 . A graph is hourglass-free if it does not have an induced subgraph isomorphic to the hourglass.

Theorem 1.1. (Broersma, Kriesell and Ryjáček, [2]) Every 4-connected hourglass-free line graph is hamiltonian connected.

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A vertex (edge, respectively) cut X of a connected graph G is *essential* if $G - X$ has at least two nontrivial components. A graph G is essentially k -connected (essentially k -edge-connected, respectively) if G does not have an essential vertex cut (essential edge cut, respectively) X with $|X| < k$. In this note, we shall improve Theorem 1.1 in the following form.

Theorem 1.2. Every 3-connected, essentially 4-connected hourglass-free line graph is hamiltonian connected.

Since adding edges will not decrease the connectivity and the essential connectivity, applying the line graph closure of Ryjáček [6], Theorem 1.2 has the following corollary.

Corollary 1.3. Every 3-connected, essentially 4-connected, claw-free graph without hourglass is hamiltonian.

2 Mechanism

Let G be a graph. An edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. This process is called *subdividing* e . For a graph G with $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e' , and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e'' . Then,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

For vertices $u, v \in V(G)$, an (u, v) -*trail* is a trail that starts with u and ends with v .

Theorem 2.1. ([4]) If G is essentially 4-edge-connected such that for every vertex $v \in V(G)$ of degree 3, G has a cycle of length at most 3 containing v , then for every pair of edges $e', e'' \in E(G)$, $G(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail.

Let G be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The **core** of this graph G , denoted by G_0 , is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2. ([5], [7]) Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G . If $\forall e', e'' \in E(G_0)$, $G(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail, then $L(G)$ is hamiltonian connected.

3 Proof of Theorem 1.2

Let G_0 be the core of G . Then as $\kappa(L(G)) \geq 3$, $\kappa'(G_0) \geq 3$. Let X be an essential edge cut of G_0 . Suppose that $|X| \leq 3$. If one side of $G_0 - X$ has only one edge, then by $\delta(G_0) \geq 3$, we must have $|X| \geq 4$, a contradiction. Therefore, both

sides of $G - X$ must have a pair of adjacent edges, and so X corresponds to an essential vertex cut of $L(G)$. Since $L(G)$ is assumed to be essentially 4-connected, $|X| \geq 4$, a contradiction. Thus we proved the claim:

(3.1) G_0 is essentially 4-edge-connected.

Fig. 1. $G[\{e_1, e_2, e_3, f_1, f_2\}]$ and $L(G)[\{e_1, e_2, e_3, f_1, f_2\}]$

We shall prove the next claim:

(3.2) $\forall v \in D_3(G_0)$, G_0 has a cycle of length at most 3 intersecting $E_{G_0}(v)$.

By contradiction. Let $v \in D_3(G_0)$ with $E_{G_0}(v) = \{e_1, e_2, e_3\}$ such that no edge in $E_{G_0}(v)$ lies in a cycle of length at most 3. Let v_1, v_2, v_3 denote the vertices of G_0 adjacent to v such that $e_i = vv_i$, ($1 \leq i \leq 3$). Since $\delta(G_0) \geq 3$, we can assume that $\{f_1, f_2\} \subseteq E_{G_0}(v_3) - \{e_3\}$ (see Figure 1(a)). By the definition of a core, either $e_3 \in E(G)$ or e_3 is a new edge replacing a path of length 2 in G (see Figure 2).

Fig. 2. The edge cut $\{e'_1, e_2, e'_3\}$ in G

If $e_3 \in E(G)$, and if neither e_1 nor e_2 is adjacent to f_1 or f_2 (see Figure 1(a) and (b)), then $L(G)$ would have an induced hourglass, contrary to the assumption that $L(G)$ is hourglass-free. Hence we may assume that $e_1 f_1 \in E(L(G))$ (see Figure 1(c)). Then $f_1 \in E_{G_0}(v_1)$, and so $G_0[\{e_1, e_3, f_1\}] \cong K_3$, contrary to the choice of v .

By symmetry, we may assume that $\forall i \in \{1, 2, 3\}$, e_i is a new edge which replaces a path with edges $\{e_i, e'_i\}$ in G , where we also use e_i to denote the edge adjacent to v in G . Then $\{e'_1, e_2, e'_3\}$ corresponds to an essential vertex

cut of $L(G)$ (see Figure 2), contrary to the assumption that $L(G)$ is essentially 4-connected. This proves Claim (3.2).

By (3.1), (3.2) and by Theorem 2.1, $\forall e', e'' \in E(G_0)$, $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail, and so by Lemma 2.2, $L(G)$ is hamiltonian connected.

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